



Multiple positive solutions for some P-Laplacian nonlinear problem at infinity

Robabeh Sahandi Torogh

Department of Mathematics, Varamin-Pishva branch, Islamic Azad University
Varamin, Iran

sahandi@iauvaramin.ac.ir

(Received July 2017 ; Published Sep 2017)

ABSTRACT

In recent years, boundary value problem of second-order have received a lot of attention. In this paper, I study the existence of positive solutions for a class of p-Laplacian boundary value problem at infinity. The fixed point theorems in cones is the our main tools to prove the existence of solutions. I provide sufficient conditions under which this system has solution. I establish some propositions to prove the existence of positive solutions for these equations.

Keywords: Boundary value problem, Positive solution, (p,q)-Laplacian system, Fixed point

DOI:10.14331/ijfps.2017.330105

INTRODUCTION

In recent years, boundary value problems have received a lot of attention. For example (Liang & Zhang, 2009; Pang, Lian, & Ge, 2007) have studied the existence of positive solutions for some boundary value problems.

In this paper, we study the existence of positive solutions for the following system:

$$\begin{cases} (\phi_p(u'))' + m(t)f(u, v) = 0 \\ (\phi_p(v'))' + n(t)g(u, v) = 0 \end{cases} \quad (1)$$

$$\begin{cases} u(0) - \alpha_0(u'(\eta)) = 0, \quad u'(+\infty) = 0, \\ v(0) - \beta_0(v'(\xi)) = 0, \quad v'(+\infty) = 0 \end{cases} \quad (2)$$

Where

$$\phi_p(s) = |s|^{p-2}s, \quad p > 1, \quad \phi_q = (\phi_p)^{-1}, \quad \frac{1}{p} + \frac{1}{q} = 1,$$

$\eta, \xi \in (0, +\infty), m, n : [0, +\infty) \rightarrow [0, +\infty)$ have countably many singularities on $[0, +\infty)$. α_0, β_0 are functions which satisfy the conditions that there are nonnegative $\alpha_1, \beta_1, \alpha_2, \beta_2$ such that

$$\alpha_1 x \leq \alpha_0(x) \leq \alpha_2 x, \quad \beta_1 x \leq \beta_0(x) \leq \beta_2 x \quad \text{for } x, y \in \mathbb{R}.$$

Liang & Zhang, (2009), studied the existence of positive solutions for

$$\begin{cases} (\phi(u'(t)))' + a(t)f(t, u(t)) = 0, 0 \in [0, +\infty) \\ u(0) - B_0 u'(\eta) = 0, \quad u'(+\infty) = 0 \end{cases}$$

Where $\phi(s) : \mathbb{R} \rightarrow \mathbb{R}$ is an increasing homomorphism and $\phi(0) = 0$. $\eta \in (0, +\infty), a : [0, +\infty) \rightarrow [0, +\infty)$.

Now, we assume that the following conditions:

$H_1)$ $f, g \in C([0, +\infty)^2, [0, +\infty))$, $f(0,0) \neq 0, g(0,0) \neq 0$ on any subinterval of $[0, +\infty)$ and when u, v are bounded, $f((1+t)u, (1+t)v), g((1+t)u, (1+t)v)$ are bounded on $[0, +\infty)^2$.

$H_2)$ There exists a sequence $\{t_i\}_{i=1}^{\infty}$ such that $1 \leq t_{i+1} \leq t_i$, $\lim_{i \rightarrow +\infty} t_i = t_0 < \infty$, $t_0 > 1$, $\lim_{t \rightarrow t_i} m(t) = \infty$, $i = 1, 2, \dots$, and

$$\int_0^{+\infty} \phi_p^{-1} \left(\int_s^{+\infty} m(t) dt \right) ds < +\infty$$

$$\int_0^{+\infty} \phi_p^{-1} \left(\int_s^{+\infty} n(t) dt \right) ds < +\infty,$$

(3)

H_3) There exists a sequence $\{t_i\}_{i=1}^\infty$ such that $0 < t_{i+1} < t_i < 1$, $\lim_{i \rightarrow +\infty} t_i = t_0 < \infty$, $t_0 > 1$, $\lim_{t \rightarrow t_i} a(t) = \infty$, $i = 1, 2, \dots$ and (3) holds.

SOME DEFINITIONS AND FIXED POIN THEOREMS

Definition (1)

Let $(X, \|\cdot\|)$ be a real Banach space and a non-empty, closed, convex C subset of X is called a Cone of X , If it satisfies the following conditions:

i) If $x \in C$ and $\lambda \geq 0$ implies that $\lambda x \in C$, ii) If $x \in C$ and $-x \in C$ implies that $x = 0$,

Every cone C subset of X includes an ordering in X which is given by $x \leq y$ if and only if $y - x \in C$.

Definition (2)

A map $\psi: P \rightarrow [0, +\infty)$ is called nonnegative continuous concave functional provided ψ is nonnegative, continuous and satisfies,

$$\psi(tx + (1 - t)y) \geq t\psi(x) + (1 - t)\psi(y)$$

for all $x, y \in P$ and $t \in [0, 1]$.

Similarly, we say the map β is a nonnegative continuous convex functional on a cone P of X $\beta: P \rightarrow [0, +\infty)$ is continuous and $\beta(tx + (1 - t)y) \geq t\beta(x) + (1 - t)\beta(y)$ for all $x, y \in P$ and $t \in [0, 1]$. The main tool of this paper is the following fixed-point:

Theorem (3) (Deimling, 2010) .

Let E be a Banach space and P subset of E be a cone in E . let $r > 0$ define $\Omega_r = \{x \in P \mid \|x\| < r\}$. Assume that $T: P \cap \overline{\Omega_r} \rightarrow P$ is completely continuous operator such that $Tx \neq x$ for $x \in \partial\Omega_r$,

- i) If $\|Tu\| \leq \|u\|$ for $u \in \partial\Omega_r$ then $i(T, \Omega_r, P) = 1$
- ii) If $\|Tu\| \geq \|u\|$ for $u \in \partial\Omega_r$ then $i(T, \Omega_r, P) = 0$

PRELIMINARIES AND LEMMAS

Let,

$$E = \left\{ (u, v) \in c[0, +\infty) \times c[0, +\infty) \mid \sup_{0 \leq t} \frac{|u(t)|}{1+t} < \infty, \sup_{0 \leq t} \frac{|v(t)|}{1+t} < \infty \right\}$$

Then E is a banach space with the norm $\|(u, v)\| = \|u\| + \|v\|$ where $\|u\| = \sup_{0 \leq t} \frac{|u(t)|}{1+t} < +\infty$.

Define cone K subset of E by

$$K = \left\{ (u, v) \in E \mid u, v \text{ are concaves, } \lim_{t \rightarrow \infty} u'(t) = 0, \lim_{t \rightarrow \infty} v'(t) = 0, \right\}$$

Lemma (4) (Liang & Zhang, 2009).

Suppose H_2 holds. Then for any $\gamma \in (1, +\infty)$ which satisfies $0 < \int_{\frac{1}{\gamma}}^{\gamma} m(t)dt < +\infty$, $0 < \int_{\frac{1}{\gamma}}^{\gamma} n(t)dt < +\infty$, and the functions

$$K_1(t) = \int_{s_0}^t \phi_p^{-1} \left(\int_s^{s_0} m(\rho) d\rho \right) ds + \alpha_1 \phi_p^{-1} \left(\int_t^{s_0} n(\rho) d\rho \right)$$

$$K_2(t) = \int_{s_0}^t \phi_p^{-1} \left(\int_s^{s_0} n(\rho) d\rho \right) ds + \beta_1 \phi_p^{-1} \left(\int_t^{s_0} m(\rho) d\rho \right)$$

Are continuous and positive on $[\frac{1}{s_0}, s_0]$.

In addition $H_i = \min_{t \in [\frac{1}{s_0}, s_0]} K_i(t) > 0$, $i = 1, 2$.

Lemma (5) (Liang & Zhang, 2009).

Let u be a nonnegative concave function on $[0, +\infty)$ and $\lim_{t \rightarrow \infty} u'(t) = 0$, and $[a, b]$ be a subset of $(0, +\infty)$.

Then $u(t) \geq \lambda(t)\|u\|$ where $\lambda(t) = \begin{cases} \sigma, & t \geq \sigma \\ \frac{|u(t)|}{1+t}, & t \leq \sigma \end{cases}$ and $\sigma = \inf \{ \xi \in [0, +\infty) : \sup_{0 \leq t < +\infty} \frac{|u(t)|}{1+t} = \frac{|u(\xi)|}{1+\xi} \}$

Now, we define an operator

$$T: K \rightarrow c[0, +\infty) \times c[0, +\infty)$$

$$T(u, v)(t) = (T_1(u, v), T_2(u, v))(t)$$

Such that

$$T_1(u, v) = \int_0^t \phi_p^{-1} \left(\int_s^{+\infty} m(\rho) f(u(\rho), v(\rho)) d\rho \right) ds + \alpha_0 \phi_p^{-1} \left(\int_t^{s_0} m(\rho) f(u(\rho), v(\rho)) d\rho \right), \tag{4}$$

$$T_2(u, v) = \int_0^t \phi_p^{-1} \left(\int_s^{+\infty} n(\rho) g(u(\rho), v(\rho)) d\rho \right) ds + \beta_0 \phi_p^{-1} \left(\int_t^{s_0} n(\rho) g(u(\rho), v(\rho)) d\rho \right), \tag{5}$$

Lemma (6) (Liu, 2003)

Let W be a bounded subset of K . Then W is relatively compact in E if $\left\{ \frac{w(t)}{1+t} \right\}$ are equicontinuous on any finite subinterval of $[0, +\infty)$ and for any $\varepsilon > 0$ there exists $N > 0$ such that $\left| \frac{x(t_1)}{1+t_1} - \frac{x(t_2)}{1+t_2} \right| < \varepsilon$, uniformly with respect to $x \in W$ as $t_1, t_2 \geq N$, where $W(t) = \{x(t) : x \in W\}$, $t \in [0, +\infty)$.

Lemma (7)

Let H_1, H_2, H_3 hold. Then $T: K \rightarrow K$ is completely continuous.

MAIN RESULT

Theorem (8)

Suppose that H_1, H_2, H_3 hold. Let $\{\lambda_k\}_{k=1}^{+\infty}$ such that $\lambda_k \in (t_k, t_{k+1})$, $k = 1, 2, \dots$. Let $\{m_k\}_{k=1}^{+\infty}$ and $\{M_k\}_{k=1}^{+\infty}$ be such that $M_{k+1} < \frac{\Gamma(\frac{1}{\lambda_k})}{1+\lambda_k} m_k < m_k < qm_k < M_k$, and for $k \in N$, we assume that f, g satisfy,

H₄ $f((1+t)u, (1+t)v) \geq \phi_p(qm_k), g((1+t)u, (1+t)v) \geq \phi_p(qm_k)$

For

$$(t, u, v) \in \left[\frac{1}{\lambda_k}, \lambda_k\right] \times \left[\frac{\Gamma\left(\frac{1}{\lambda_k}\right)}{1+\lambda_k} m_k, m_k\right] \times \left[\frac{\Gamma\left(\frac{1}{\lambda_k}\right)}{1+\lambda_k} m_k, m_k\right].$$

H₅ $f((1+t)u, (1+t)v) \geq \phi_p(QM_k), g((1+t)u, (1+t)v) \geq \phi_p(QM_k),$

For $(t, u, v) \in [0, +\infty] \times [0, M_k] \times [0, M_k]$

where

$$q \in (\Gamma_1, +\infty), Q \in (0, \Gamma_2), \quad \Gamma_1 = \frac{1+t_0}{L}, L > 0,$$

$$\Gamma_2 = \frac{1}{\max(\phi_p^{-1}(\int_0^{+\infty} m(\rho)d\rho)(1+\alpha_2), \phi_p^{-1}(\int_0^{+\infty} n(\rho)d\rho)(1+\beta_2))}$$

Then the boundary value system (1) and (2) has infinitely many solutions $\{(u_k, v_k)\}_{k=1}^{+\infty}$ such that $m_k \leq \|(u_k, v_k)\| \leq M_k, k = 1, 2, \dots$

Proof. We assume that the sequence $\{\Omega_{1k}\}_{k=1}^{+\infty}$ and $\{\Omega_{2k}\}_{k=1}^{+\infty}$ of open subsets of E be as following:

$$\Omega_{1k} = \{(u, v) \in K \mid \|(u, v)\| < 2m_k\},$$

$$\Omega_{2k} = \{(u, v) \in K \mid \|(u, v)\| < 2M_k\}, \quad k = 1, 2, \dots$$

We know that $1 < t_0 \leq t_{k+1} < \lambda_k < t_k < +\infty, k = 1, 2, \dots$, so from lemma (5) for $k \in N$ and $u, v \in K$ we have $u(t) \geq \Gamma(t)\|u\|, t \in \left[\frac{1}{\lambda_k}, \lambda_k\right]$.

Let $k \in N$ and $(u, v) \in \partial\Omega_{1k}$, then we have

$$2m_k = \|(u, v)\| = \sup_{t \geq 0} \frac{|u(t)|}{1+t} + \sup_{t \geq 0} \frac{|v(t)|}{1+t} \geq \frac{|u(\frac{1}{\lambda_k})|}{1+\lambda_k} + \frac{|v(\frac{1}{\lambda_k})|}{1+\lambda_k} \geq \frac{\Gamma\left(\frac{1}{\lambda_k}\right)}{1+\lambda_k} (\|(u, v)\|), \quad t \in \left[\frac{1}{\lambda_k}, \lambda_k\right].$$

From (H₄) we have $f((1+t)u, (1+t)v) \geq \phi_{p_1}(qm_k)$, we know that $\left(\frac{1}{t_0}, t_0\right) \subseteq \left[\frac{1}{\lambda_k}, \lambda_k\right]$, if (H₂) holds, we consider three cases:

i) If $\eta \in \left[\frac{1}{t_0}, t_0\right]$: we have:

$$\begin{aligned} \|T_1(u, v)\| &= \sup_{t \geq 0} \frac{1}{1+t} \left| \int_0^t \phi_p^{-1} \left(\int_s^{+\infty} m(\rho) f(u(\rho), v(\rho)) d\rho \right) ds + \alpha_0 \phi_p^{-1} \left(\int_\eta^{+\infty} m(\rho) f(u(\rho), v(\rho)) d\rho \right) \right| \\ &\geq \frac{1}{1+t_0} (qm_k) \int_{\frac{1}{t_0}}^\eta \phi_p^{-1} \left(\int_s^{t_0} m(\rho) d\rho \right) ds + \alpha_1 \phi_p^{-1} \left(\int_\eta^{t_0} m(\rho) d\rho \right) \end{aligned}$$

$$= \frac{qm_k}{1+t_0} K_1(\eta) > \frac{Lqm_k}{1+t_0} > 2m_k = \|(u, v)\|.$$

ii) If $\eta \in (0, \frac{1}{t_0})$ from (4) and (H₄) and lemma (4) we see :

$$\begin{aligned} \|T_1(u, v)\| &= \sup_{t \geq 0} \frac{1}{1+t} \left| \int_0^t \phi_p^{-1} \left(\int_s^{+\infty} m(\rho) f(u(\rho), v(\rho)) d\rho \right) ds + \alpha_0 \phi_p^{-1} \left(\int_\eta^{+\infty} m(\rho) f(u(\rho), v(\rho)) d\rho \right) \right| \\ &\leq \sup_{t \geq 0} \frac{1}{1+t} \alpha_1 \phi_p^{-1} \left(\int_{\frac{1}{t_0}}^{t_0} m(\rho) f(u(\rho), v(\rho)) d\rho \right) \\ &\geq \frac{qm_k}{1+t_0} \alpha_1 \phi_p^{-1} \left(\int_{\frac{1}{t_0}}^{t_0} m(\rho) d\rho \right) \\ &= \frac{qm_k}{1+t_0} K_1\left(\frac{1}{t_0}\right) > \frac{Lqm_k}{1+t_0} > 2m_k = \|(u, v)\|. \end{aligned}$$

iii) If $\eta \in (t_0, +\infty)$. From (4), (H₄) and lemma (4) we have

$$\begin{aligned} \|T_1(u, v)\| &= \sup_{t \geq 0} \frac{1}{1+t} \left| \int_0^t \phi_p^{-1} \left(\int_s^{+\infty} m(\rho) f(u(\rho), v(\rho)) d\rho \right) ds + \alpha_0 \phi_p^{-1} \left(\int_\eta^{+\infty} m(\rho) f(u(\rho), v(\rho)) d\rho \right) \right| \\ &\geq \frac{qm_k}{1+t_0} K_1(t_0) > \frac{Lqm_k}{1+t_0} > 2m_k = \|(u, v)\|. \end{aligned}$$

Since

$$\|T(u, v)\| = \|T_1(u, v)\| + \|T_2(u, v)\| \geq \|(u, v)\|,$$

so from theorem (3) implies that

$$i(T, \Omega_{1k}, K) = 0, \tag{6}$$

Suppose that $(u, v) \in \partial\Omega_{2k}$ and $u, v \in [0, M_k]$. Thus

$$\frac{u(t)}{1+t} \leq \sup_{t \geq 0} \frac{|u(t)|}{1+t} \leq \|u\| = M_k,$$

$$\frac{v(t)}{1+t} \leq \sup_{t \geq 0} \frac{|v(t)|}{1+t} \leq \|v\| = M_k$$

From (H₄) we have

$$f((1+t)u, (1+t)v) \leq \phi_p(QM_k),$$

so

$$\begin{aligned} \|T_1(u, v)\| &= \sup_{t \geq 0} \frac{1}{1+t} \left| \int_0^t \phi_p^{-1} \left(\int_s^{+\infty} m(\rho) f(u(\rho), v(\rho)) d\rho \right) ds + \alpha_0 \phi_p^{-1} \left(\int_\eta^{+\infty} m(\rho) f(u(\rho), v(\rho)) d\rho \right) \right| \\ &\leq (1 + \alpha_2) \phi_p^{-1} \left(\int_0^{+\infty} m(\rho) f(u(\rho), v(\rho)) d\rho \right) \leq QM_k (1 + \alpha_2) \phi_p^{-1} \left(\int_0^{+\infty} m(\rho) d\rho \right) \leq M_k = \|u\| = \frac{1}{2} \|(u, v)\|. \end{aligned}$$

Similarly we can see $\|T_2(u, v)\| \leq \|v\| = \frac{1}{2}\|(u, v)\|$.

Then

$$\|T(u, v)\| = \|T_1(u, v)\| + \|T_2(u, v)\| \leq \|(u, v)\|$$

for $(u, v) \in \partial\Omega_{2k}$, Hence, from theorem (3) we have

$$i(T, \Omega_{2k}, K) = 1, \quad (7)$$

Thus from additivity the fixed-point index we have

$$i(T, \Omega_{2k} \setminus \overline{\Omega_{1k}}, K) = 1$$

and T has a fixed point in $\Omega_{2k} \setminus \overline{\Omega_{1k}}$ such that

$$m_k \leq \|(u, v)\| \leq M_k, \text{ for } k \in N$$

REFERENCES

- Deimling, K. (2010). *Nonlinear functional analysis*: Courier Corporation.
- Liang, S., & Zhang, J. (2009). The existence of countably many positive solutions for some nonlinear three-point boundary problems on the half-line. *Nonlinear Analysis: Theory, Methods & Applications*, 70(9), 3127-3139.
- Liu, Y. (2003). Existence and unboundedness of positive solutions for singular boundary value problems on half-line. *Applied Mathematics and Computation*, 144(2), 543-556.
- Pang, H., Lian, H., & Ge, W. (2007). Multiple positive solutions for second-order four-point boundary value problem. *Computers & Mathematics with Applications*, 54(9), 1267-1275.