Existence of Positive Solutions For a Class of Multi-Point P-Laplacian Nonlinear System

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ABSTRACT

In this paper the existence of positive solutions for a class of p-Laplacian boundary value system is studied. In recent years, boundary value problem have received a lot of attention. The fixed point theorems in cones is our main tools to prove the existence of solutions. I provide sufficient conditions under which these systems has positive solutions. I establish some propositions to prove the existence of positive solutions for these equation.

Keywords: Fixed point theorem; p-Laplacian operator; positive solutions

INTRODUCTION

In this paper, we prove the existence of positive solutions for the following system,

\[
\begin{align*}
\left( (\phi_p(u'))' + f(t, u, v) = 0 \\
(\phi_p(v'))' + g(t, u, v) = 0
\end{align*}
\]

For \( \sum_{i=1}^{n} \alpha_i u(\xi_i) = 0 \) \( \sum_{i=1}^{n} \beta_i u(\xi_i) = 0 \) \( \sum_{i=1}^{n} \gamma_i v(\xi_i) = 0 \) \( \sum_{i=1}^{n} \delta_i v(\xi_i) = 0 \)

Where

\[
\phi_p(s) = |s|^{p-2}s, \quad p > 1, \quad \phi_q = (\phi_p)^{-1}, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad \xi_i \in (0,1)
\]

with \( 0 < \xi_1 < \xi_2 < \ldots < \xi_n < 1 \).

For \( a_i, b_i, \gamma_i, \delta_i \in [0, +\infty) \) and

\[
0 < \sum_{i=1}^{n} a_i, \sum_{i=1}^{n} b_i, \sum_{i=1}^{n} \gamma_i, \sum_{i=1}^{n} \delta_i < 1,
\]

where \( f, g \in C([0,1] \times [0, +\infty) \times [0, +\infty), (-\infty, +\infty)) \)

In recent years, BVP have received a lot of attention. There are many papers concerned with the p-Laplacian equations. For example (Liang & Zhang, 2009; Pang, Lian, & Ge, 2007; Sun & Ge, 2007) have studied the existence of positive solutions for some boundary value problems. (Harjani, López, & Sadarangani, 2011) have studied the fixed point theorems in metric spaces. The existence of multiple positive solutions to the boundary value problem was studied by (Ji, Feng, & Ge, 2008).

DEFINITIONS

Definition (1) Let \( (X, \| \|) \) be a real Banach space and a non-empty, closed, convex \( C \) subset of \( X \) is called a Cone of \( X \). If it satisfies the following conditions: i) If \( x \in C \) and \( \lambda \geq 0 \) implies that \( \lambda x \in C \). ii) If \( x \in C \) and \( -x \in C \) implies that \( x = 0 \). Every cone \( C \) subset of \( X \) includes an ordering in \( X \) which is given by \( x \leq y \) if and only if \( y - x \in C \).

Definition (2) A map \( \psi: P \to [0, +\infty) \) is called nonnegative continuous concave functional provided \( \psi \) is nonnegative, continuous and satisfies \( \psi(tx + (1-t)y) \geq \)
\( t \psi(x) + (1 - t) \psi(y) \) for all \( x, y \in P \) and \( t \in [0, 1] \). Similarly, we say the map \( \beta \) is a nonnegative continuous convex functional on a cone \( P \) of \( \mathbb{R}^n \) is continuous and \( \beta(x + (1 - t)y) \geq t \beta(x) + (1 - t) \beta(y) \) for all \( x, y \in P \) and \( t \in [0, 1] \).

**Definition (3).** An operator is called completely continuous if it is continuous and maps bounded sets into pre-compact sets. The main tool of this paper is the following lemma.

**Lemma (4) (Lan, 2001)** Let \( B \) be an open bounded subset of \( X \) with \( B_k = B \cap K \neq \emptyset \). Assume that \( T: B_k \rightarrow K \) is completely continuous operator such that \( Tx \neq x \) for \( x \in \partial B_k \), then the following results hold:

i) If \( \|Tx\| \leq \|x\| \) for \( x \in \partial \Omega \), then \( i_k(T, B_k) = 1 \)

ii) If there exists \( a \in K \setminus \{0\} \) such that \( x \neq Tx + \lambda a \) for \( u \in \partial B_k \) then \( i_k(T, B_k) = 0 \)

iii) Let \( U \) be an open subset of \( X \) such that \( U \subset B_k \). If \( i_k(T, U) = 0 \) and \( i_k(T, U_k) = 0 \), then \( T \) has a fixed point in \( B_k \setminus U \).

The same result holds if \( i_k(T, B_k) = 0 \) and \( i_k(T, U_k) = 1 \).

**PRELIMINARIES AND LEMMAS**

Let \( E = c[0,1] \times c[0,1] \). Then \( E \) is a Banach space with the norm \( \|(u,v)\| = \|u\| + \|v\| \) where \( \|u\| = \max_{t \in [0,1]} |u(t)| \). Define the cone \( K \) subset of \( E \) by \( K = \{(u,v) \in E \mid u \) and \( v \) are concaves on \([0,1], u(t) \geq 0, v(t) \geq t \} \).

By assumption we have that \( H_2, H_4 \) hold.

**Lemma (5) (Ji et al., 2008)** Suppose \( h(t, m(t), n(t)) > 0 \) for \( m(t) \geq 0, n(t) \geq 0 \), then for \( m, n \in C^+ [0,1] \), the problem

\[
\begin{aligned}
& \left( \phi_p(u') \right) + h(t, m(t), n(t)) = 0, \quad t \in (0,1) \\
& u(0) = \sum_{i=1}^{n} \alpha_i u(\xi_i), \quad u(1) = \sum_{i=1}^{n} \beta_i u(\xi_i)
\end{aligned}
\]

(2)

Has a solution

\[
u(t) = \sum_{i=1}^{n} \alpha_i \int_0^1 \phi_q(M_{n,m} - \int_0^s h(r, m(r), n(r))dr)ds \] + \sum_{i=1}^{n} \beta_i \int_0^1 \phi_q(M_{m,n} - \int_0^s h(r, m(r), n(r))dr)ds \]

Where \( M_{n,m} \) satisfies

\[
\sum_{i=1}^{n} \alpha_i \left( 1 - \sum_{i=1}^{n} \beta_i \right) \int_0^1 \phi_q(M_{n,m} - h(r, m(r), n(r))dr)ds \] + \sum_{i=1}^{n} \beta_i \int_0^1 \phi_q(M_{m,n} - h(r, m(r), n(r))dr)ds \]

(3)

Then there exists a unique \( M_{n,m} \in \left( 0, \int_0^1 h(t, m(t), n(t))dt \right) \) satisfying (3). This implies that there is a unique \( \Delta \) such that

\[
M_{m,n} = \int_0^\Delta h(t, m(t), n(t))dt
\]

**Proof.** We define for \( h(t, m(t), n(t)) > 0 \)

\[
N_{m,n}(l) = \sum_{i=1}^{n} \alpha_i \left( 1 - \sum_{i=1}^{n} \beta_i \right) \int_0^1 \phi_q(l) - \sum_{i=1}^{n} \beta_i \int_0^1 \phi_q(l) - \int_0^1 h(r, m(r), n(r))dr)ds + \int_0^1 \phi_q(l) - \int_0^1 h(r, m(r), n(r))dr)ds
\]

(4)

We see that \( N_{m,n}; R \rightarrow R \) is continuous and increasing. Then \( N_{m,n}(0) < 0 \) and \( N_{m,n}(\int_0^1 h(r, m(r), n(r))dr) > 0 \) and there exists a \( t \in (0,\int_0^1 h(t, m(t), n(t))dt) \) such that \( N_{m,n}(t) = 0 \). Thus there exists \( \Delta \) such that \( M_{m,n} = \int_0^\Delta h(r, m(r), n(r))dr) \).

**Lemma (6) (Ji et al., 2008)** Suppose that \( h(r, m(r), n(r)) > 0 \) for \( m, n \in C^+ [0,1] \). Then the solution of BVP (2) can also be expressed,

\[
u(t) = \sum_{i=1}^{n} \alpha_i \left( 1 - \sum_{i=1}^{n} \beta_i \right) \int_0^1 \phi_q(N_{m,n} - \int_0^s h(r, m(r), n(r))dr)ds - \int_0^1 \phi_q(N_{m,n} - \int_0^s h(r, m(r), n(r))dr)ds
\]

(5)
Then there exists a unique \( N_{m,n} \in (0, \int_0^1 h(t, m(t), n(t))\,dt) \) satisfying (5). This implies that there is a unique \( \Delta' \in (0,1) \) such that 
\[
N_{m,n} = \int_0^{\Delta'} h(r, m(r), n(r))\,dr.
\]

**Lemma (7)** (Ji et al., 2008) If \( h(r, m(r), n(r)) > 0 \) for \( m, n \in C^*[0,1] \). Then the solution \( u(t) \) of (2) has the following properties.

i) \( u(t) \) is concave on \((0,1)\) and \( u(t) \geq 0 \),

ii) there exists a unique \( t_0 \in (0,1) \) such that \( u(0) = \max_{0 \leq t \leq 1} u(t) \), \( u'(t_0) = 0 \),

iii) \( \Delta = \Delta' = t_0 \).

**Lemma (8)** (Wang & Zhang, 2006) Let \( u(t) = 0 \) is concave on \([0,1] \), \( \eta \in (0, \frac{1}{2}) \) then \( u(t) \geq \eta ||u|| \), \( t \in [\eta, 1 - \eta] \). We define \( \varphi(t) = \min \{ t, 1 - t \} \), \( t \in (0,1) \).

\[
\begin{align*}
Y_1 &= \frac{\eta \min \left\{ \int_{1/2}^{1/3} \int_{1/2}^{1/3} (1 - \int_{t_0}^s \phi(s)) \,ds \, \int_{1/2}^{1/3} (1 - \int_{t_0}^s \phi(s)) \,ds \right\}}{\int_0^{1/3} \phi(s) \,ds} \\
Y_2 &= \frac{\eta \min \left\{ \int_{1/2}^{1/3} \int_{1/2}^{1/3} (1 - \int_{t_0}^s \phi(s)) \,ds \, \int_{1/2}^{1/3} (1 - \int_{t_0}^s \phi(s)) \,ds \right\}}{\int_0^{1/3} \phi(s) \,ds}
\end{align*}
\]

\( k_p = \{(u,v) \in K | ||(u,v)|| < \rho \}, \quad k^* = \{(u,v) \in K | \rho \phi(t) < u(t) + v(t) < \rho \}, \quad \Omega_p = \{(u,v) \in K | \min_{\eta \leq t \leq 1-\eta} (u(t) + v(t)) < \gamma \rho \}, \quad \Omega_\eta = \{(u,v) \in K | ||(u,v)|| \leq \min_{\eta \leq t \leq 1-\eta} (u(t) + v(t)) < \gamma \rho \} \)

**Lemma (9)** Suppose \( \Omega_p \) has the following properties (Lan, 2001).

i) \( \Omega_p \) is open relative to \( K \),

ii) \( k_{\rho} \subseteq \Omega_p \subseteq k_{\rho} \),

iii) \( |(u,v) \in \partial \Omega_p | if \text{ and only if } \min_{\eta \leq t \leq 1-\eta} (u(t) + v(t)) = \gamma \rho \),

iv) If \( u(t) \in \partial \Omega_p \), then \( \gamma \rho \leq u(t) + v(t) \leq \rho \) for \( \eta \leq t \leq 1 - \eta \).

Now, we define,

\[
\begin{align*}
f_{\tau,r} &= \min \left\{ \frac{f(t,u,v)}{\varphi(t)} \right\} | t \in [\eta, 1-\eta], u \in [\gamma \rho, \rho] \\
g_{\tau,r} &= \min \left\{ \frac{g(t,u,v)}{\varphi(t)} \right\} | t \in [\eta, 1-\eta], u \in [\gamma \rho, \rho] \\
f_{\phi(t)} &= \max \left\{ \frac{f(t,u,v)}{\varphi(t)} \right\} | t \in [0,1], u \in [\phi(t)r, r] \\
g_{\phi(t)} &= \max \left\{ \frac{g(t,u,v)}{\varphi(t)} \right\} | t \in [0,1], u \in [\phi(t)r, r] \\
f_0 &= \max \left\{ \frac{f(t,u,v)}{\varphi(t)} \right\} | t \in [0,1], u \in [0, \rho] \\
g_0 &= \max \left\{ \frac{g(t,u,v)}{\varphi(t)} \right\} | t \in [0,1], u \in [0, \rho] \\
f_{u} &= \lim_{u \to \alpha} \max \left\{ \frac{f(t,u,v)}{u^{\alpha-1}} \right\} | (t,v) \in [0,1] \times R^+ \\
g_{u} &= \lim_{u \to \alpha} \max \left\{ \frac{g(t,u,v)}{v^{\alpha-1}} \right\} | (t,u) \in [0,1] \times R^+ \\
f_{v} &= \lim_{v \to \alpha} \max \left\{ \frac{f(t,u,v)}{u^{\alpha-1}} \right\} | (t,v) \in [0,1] \times R^+ \\
g_{v} &= \lim_{v \to \alpha} \max \left\{ \frac{g(t,u,v)}{v^{\alpha-1}} \right\} | (t,u) \in [0,1] \times R^+
\end{align*}
\]

**MAIN RESULT**

**Theorem (10)** Suppose that \( H_1 \) hold and \( f, g \) satisfy the following conditions:

\( H_2 \) Then there exist \( r_1, r_2, r_3 \in (0, +\infty) \) and \( 2r_1 < \gamma r_2 < r_2 < 2r_3 \) such that

i) \( f(t,u,v) > 0, g(t,u,v) > 0 \), \( t \in [0,1], u, v \in [r_1 \varphi(t), \infty) \)

\[
\begin{align*}
f_{\phi(t)} &< \varphi(t), f_{\gamma \rho} > \varphi(t), \quad g_{\phi(t)} < \varphi(t), g_{\gamma \rho} > \varphi(t) \\
f_{\phi(t)} &< \varphi(t), f_{\gamma \rho} > \varphi(t), \quad g_{\phi(t)} < \varphi(t), g_{\gamma \rho} > \varphi(t)
\end{align*}
\]

\( H_3 \) There exists \( r_1, r_2, r_3 \in (0, +\infty) \) and \( r_1 < r_2 < r_3 \) such that

iii) \( f(t,u,v) > 0, g(t,u,v) > 0 \), \( t \in [0,1], u, v \in [\min\{r_1, r_2 \varphi(t)\}, \infty) \)

\[
\begin{align*}
f_{\phi(t)} &< \varphi(t), f_{\gamma \rho} > \varphi(t), \quad g_{\phi(t)} < \varphi(t), g_{\gamma \rho} > \varphi(t) \\
f_{\phi(t)} &< \varphi(t), f_{\gamma \rho} > \varphi(t), \quad g_{\phi(t)} < \varphi(t), g_{\gamma \rho} > \varphi(t)
\end{align*}
\]

Then system (1.1) has two positive solutions in \( K \).

**Proof.** Suppose \( H_2 \) holds. We define

\[
\begin{align*}
\tilde{f}(t,u,v) &= \begin{cases} f(t,u,v), u \geq r_1 \varphi(t), (4.1) \\
\{ f(t,r_1 \varphi(t),v(t)), 0 \leq u < r_1 \varphi(t) \}
\end{cases} \\
\tilde{g}(t,u,v) &= \begin{cases} g(t,u,v), u \geq r_1 \varphi(t) \\
\{ g(t,u,r_1 \varphi(t)), 0 \leq u < r_1 \varphi(t) \}
\end{cases}
\end{align*}
\]

So \( \tilde{f}(t,u,v) \in C([0,1] \times [0, +\infty) \times (0, +\infty)), \tilde{g}(t,u,v) \in C([0,1] \times [0, +\infty) \times (0, +\infty)) \).
We define an operator

\[ T: K \to E, T(u, v) = (T_1(u, v), T_2(u, v)) \]

Then \((T_1(u, v))(t)\) is completely continuous. We know that

\[ \frac{\sum_{i=1}^{n} \alpha_i}{1 - \sum_{i=1}^{n} \alpha_i} \int_{0}^{t} \phi_i \left( \int_{s}^{t} f(r, u, v) \, dr \right) ds, 0 \leq t \leq a \]

\[ \frac{\sum_{i=1}^{n} \beta_i}{1 - \sum_{i=1}^{n} \beta_i} \int_{0}^{t} \phi_i \left( \int_{s}^{t} g(r, u, v) \, dr \right) ds, a \leq t \leq 1 \]

(7)

Then \( T: K \to K \) is completely continuous. We know that

\[ \phi_i(\int_{0}^{t} f(r, u, v) \, dr) \geq 0, \quad \phi_i(\int_{0}^{t} g(r, u, v) \, dr) \geq 0 \]

(8)

From lemma 7, \((T(u, v))(t) = (T_1(u, v), T_2(u, v))(t)\) is concave on \([0,1]\), for \((u, v) \in K\). Then \(TK \subset K\). The proof of \(T\) is completely continuous is similar to that (Ma, Du, & Ge, 2005). Consider modified problem:

\[ \begin{cases}
\phi_i'(p(u')) + f(t, u, v) = 0 \\
\phi_i'(p(v')) + g(t, u, v) = 0
\end{cases} \]

\[ \begin{cases}
\nu(0) = \sum_{i=1}^{n} \lambda_i \nu(\xi_i), & u(0) = \sum_{i=1}^{n} \alpha_i u(\xi_i), \\
\nu(1) = \sum_{i=1}^{n} \lambda_i \nu(\xi_i), & u(1) = \sum_{i=1}^{n} \beta_i u(\xi_i).
\end{cases} \]

By condition \(H_2\) (ii) we conclude

\[ \tilde{f}_{\nu}(p(r_1)) \leq \phi_{i}(m), \tilde{g}_{\nu}(p(r_2)) \leq \phi_{i}(m) \]

By \(H_2\) we prove that \(k_{2r_1} = 1\). For \((u, v) \in \partial k_{2r_1}\), from (7) and \(\tilde{f}_{\nu}(p(r_1)) < \phi_{i}(m)\), we conclude

\[ \frac{\alpha_i}{1 - \alpha_i} \int_{0}^{t} \phi_i \left( \int_{s}^{t} f(r, u, v) \, dr \right) ds \leq \frac{a_s}{1 - \sum_{i=1}^{n} \alpha_i} \int_{0}^{t} \phi_i \left( \int_{s}^{t} f(r, u, v) \, dr \right) ds \]

Finally, we obtain

\[ \frac{\alpha_i}{1 - \alpha_i} \int_{0}^{t} \phi_i \left( \int_{s}^{t} f(r, u, v) \, dr \right) ds \leq \frac{\alpha_i}{1 - \sum_{i=1}^{n} \alpha_i} \int_{0}^{t} \phi_i \left( \int_{s}^{t} f(r, u, v) \, dr \right) ds \]

similarly, we obtain \(\|T_2(u, v)\| \leq \frac{\alpha_i}{1 - \sum_{i=1}^{n} \alpha_i} \int_{0}^{t} \phi_i \left( \int_{s}^{t} f(r, u, v) \, dr \right) ds \]

and

\[ \frac{\alpha_i}{1 - \alpha_i} \int_{0}^{t} \phi_i \left( \int_{s}^{t} f(r, u, v) \, dr \right) ds \leq \frac{\alpha_i}{1 - \sum_{i=1}^{n} \alpha_i} \int_{0}^{t} \phi_i \left( \int_{s}^{t} f(r, u, v) \, dr \right) ds \]

From lemma (4) (i) this implies that \(k_{2r_1} = 1\). Now, we show that \(k_{2r_1} = 1\). Suppose that \(a \in \partial k_1\) and \(|a(t)| = |(a_1(t), a_2(t))| = 1, t \in [0,1]\). We prove that

\[ (u, v) \neq T(u, v) + \lambda a(t, u, v)e, \quad \lambda \geq 0 \]

Otherwise, there exists \((u_0, v_0) \in \partial k_{2r_1}, \lambda_0 \geq 0\) Such that

\[ (u_0, v_0) = T(u_0, v_0) + \lambda_0 a(t, u_0, v_0) \]

We consider two cases:

\[ \begin{cases}
\sigma_{\alpha} < \frac{1}{2} & \text{from (7) and lemma (7), for } t \in [\eta, 1 - \eta], \text{ we have,} \\
\nu(0) = \sum_{i=1}^{n} \lambda_i \nu(\xi_i), & u(0) = \sum_{i=1}^{n} \alpha_i u(\xi_i), \\
\nu(1) = \sum_{i=1}^{n} \lambda_i \nu(\xi_i), & u(1) = \sum_{i=1}^{n} \beta_i u(\xi_i).
\end{cases} \]

By \(H_2\) we prove that \(k_{2r_1} = 1\). For \((u, v) \in \partial k_{2r_1}\), from (7) and \(\tilde{f}_{\nu}(p(r_1)) < \phi_{i}(m)\), we conclude

\[ \frac{\alpha_i}{1 - \alpha_i} \int_{0}^{t} \phi_i \left( \int_{s}^{t} f(r, u, v) \, dr \right) ds \leq \frac{\alpha_i}{1 - \sum_{i=1}^{n} \alpha_i} \int_{0}^{t} \phi_i \left( \int_{s}^{t} f(r, u, v) \, dr \right) ds \]

so we conclude that \(\gamma r_2 > \gamma r_2 + \lambda_0\).
this implies that $\gamma r_2 > \gamma r_2 + \lambda_0$. From these two cases we conclude $\gamma r_2 > \gamma r_2 + \lambda_0$ which is a contradiction. Then from lemma (4) (ii), we implies that $i_k(T,\Omega_{r_2}) = 0$. Similarly we can prove that $i_k(T,K_{-2 r_2}) = 1$. Lemma (4) implies that the problem (8) has three positive solutions such that $(u_1,v_1) \in K_{-2 r_2}, (u_2,v_2) \in \Omega_{r_2} \setminus K_2 r_2, (u_3,v_3) \in K_{-2 r_2}$. It is clear that these solutions belong to $[2r_2 \varphi(t),+\infty)$ and satisfy the system (1).


