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Leaky Mode Analysis of Non-Homogeneous Asymmetric Slab Waveguides

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ABSTRACT

In general, leaky mode analysis of the waveguides is done by calculating the propagation constant from their eigenvalue equation. This equation includes the dispersion effect in waveguide too. In situations in which refractive index in the guiding region is not constant, the eigenvalue equation can be solved by using different methods. The eigenvalue equation of an asymmetric slab waveguide, with non-homogeneous refractive index of guiding layer, and its field profile, are solved. The research method is based on the differential transfer matrix method (DTMM). The resulting equation is a nonlinear equation containing the dispersion effect. In this non-homogeneous waveguide, the initial solution is obtained from WKB method. It can be used in solving the waveguide eigenvalue equation based on the Newton-Raphson method for better accuracy. The results of WKB method are compared with the answers given by the equation.

Key words: Differential transfer matrix method (DTMM), dispersion relation, leaky modes, non-homogeneous slab waveguide

INTRODUCTION

Calculating the field distribution in any waveguide, considering the effects of radiated modes and their integral relations, is cumbersome (Lee, Chung, Coldren, & Dagli, 1995) . Results of the waveguide analysis, includes two discrete and continuous spectra which are related to propagating and radiating modes waveguide main modes, respectively. To have a continuous spectrum many samples of these modes are needed. In many applications this continuous spectrum can be substituted by integrating a set of discrete leaky modes. This is due to the less need for leaky mode samples in mode expansion. Thus, it simplifies the analysis of the waveguides.

The analysis of leaky mode has been studied in many practical devices such as curved waveguides (Thayagarajan, Shenoy, & Ghatak, 1978), waveguide lasers (Stetten & Kneubuhl, 1968), anti-guided laser arrays (Botez, Mawst, & Peterson, 1988), polarizer and coplanar waveguides (Thyagarajan, Diggavi, & Ghatak, 1978). According to these applications; a large amount of research on leaky modes with different methods has been done (Ye & Yevick, 2001; Khorasani & Mehrany, 2003; Zhu & Lu, Leaky modes of slab waveguides—Asymptotic solutions, 2006; Zhu, Chen, & Tang, Leaky modes of optical waveguides with varied refractive index for microchip optical interconnect applications— Asymptotic solutions, 2008; Hu & Menyuk, 2009; Jiani, Khorasani, Rashidian, & Mohammdi, 2009; Stowell & Tausch, 2009). These works are mostly related to multi-layer structures and fibers. Leaky waves, as complex solutions of eigenvalue equation, which essentially are related to propagation constant in dielectric waveguides play an important role in many electromagnetic (EM) analysis tools .When the reflective index of the middle layer is constant (Zhu & Lu, Leaky modes of slab waveguides—Asymptotic solutions, 2006), this equation, which includes dispersion effect too, is obtained by using boundary conditions and field profiles in each layer. But, if the reflective index of the middle layer is non-homogeneous and the waveguide is asymmetric, the analysis will be complicated. In this paper, other approximation or numerical methods such as WKB is used (Zhu & Lu, Leaky modes of slab waveguides—Asymptotic solutions, 2006). A non-linear equation is derived by this method which is nearly compatible with the exact method. But, in many practical problems an analytical solution, utilizing simplification of differential equations, is not possible and various numerical methods must be used (Ye & Yevick, 2001; Zhu, Chen, & Tang, Leaky modes of optical waveguides with varied refractive index for microchip optical interconnect applications— Asymptotic solutions, 2008; Stowell & Tausch, 2009).

In this research, we used a method called differential transfer matrix method (DTMM); which gives an analytical solution for leaky modes for an asymmetric nonhomogeneous waveguide. It is shown that, this method is capable to solve the homogenous linear differential equations with variable indices. The exact characteristic of this method is its extreme stability and its application in optic studies (Jiani, Khorasani, Rashidian, & Mohammdi, 2009).

The basic formulation for eigenvalue problem in TE case is presented in Sec. 2. After describing the transfer matrix method (TMM) method and its extension to DTMM method, dispersion relation of TE mode is obtained. The solution of this relation using Newton-Raphson method is presented in Sec. 3. Sec. 4 gives some numerical examples of an asymmetric non-homogeneous waveguide. Finally, we get our simulation results in Sec. 5

FORMULATION OF EIGENVALUE PROBLEM FOR TE MODE

Consider plane wave propagations, independent of y-direction, and substitute them in Maxwell equations the following relation for TE mode will be obtained (Hu & Menyuk, 2009):

$$\frac{\partial}{\partial x}\left(\frac{\partial E_y}{\partial x}\right) + (k_0{}^2 - \beta^2)E_y = 0 \qquad (1)$$

Define the y-component of magnetic field as $H_y = \phi(x)e^{i(\beta z-\omega t)}$, when ϕ is the mode profile, β is the propagation constant and ω is the angular frequency. For TE mode eigenvalue problem we have:

$$\frac{\partial^2\phi}{\partial x^2} + k_0{}^2 n^2(x)\phi = \beta^2\phi, \quad 0 < x < d \qquad (2)$$

Where refractive index function $n(x)$ is:

$$n(x) = \begin{cases} n_c, & x < 0 \\ n_0(x), & 0 < x < d \\ n_s, & x > 0 \end{cases} \qquad (3)$$

As shown in Fig. 1, the field distribution along the positive and negative transverse directions has an increased exponential form, because the studied modes are leaky modes. For a constant refractive index in these two regions, we have:

$$\frac{\partial\varphi}{\partial x} = -i\sqrt{k_0{}^2 n_c{}^2 - \beta^2}\ , \quad x \le 0$$

$$\frac{\partial\varphi}{\partial x} = i\sqrt{k_0{}^2 n_s{}^2 - \beta^2}\ , \quad x \ge d$$

Regarding to continuity of tangential components of the electric and magnetic fields in $x=0$ and $x=1$ we have:

$$\phi(0) = \phi(0^-), \phi'(0) = \phi'(0^+) \qquad (4)$$
$$\phi(d) = \phi(d^+), \phi(d) = \phi(d^+)$$

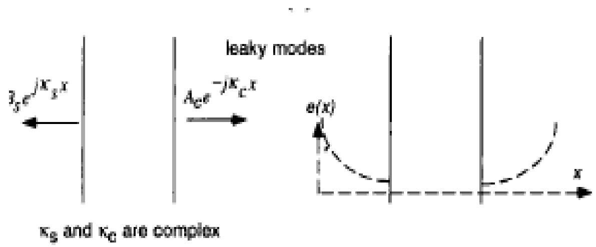


Fig 1: Field distribution of leaky modes (Lee, Chung, Coldren, & Dagli, 1995).

Assuming $\gamma_c(\beta) \equiv \sqrt{k_0{}^2 n_c^2 - \beta^2}$ and $\gamma_s(\beta) \equiv \sqrt{k_0{}^2 n_s^2 - \beta^2}$, the eigenvalue problem for TE mode is:

$$\frac{\partial^2\phi}{\partial x} + k_0{}^2 n_0{}^2(x)\phi = \beta^2\phi, \quad 0 < x < d. \qquad (5)$$

$$\phi'(x) = -i\gamma_c(\beta)\phi \qquad x = 0 \qquad (6)$$

$$\phi'(x) = i\gamma_s(\beta)\phi \qquad x = d \qquad (7)$$

Solving these non-linear equations is so difficult, but, it is possible to reduce these problems to one non-linear algebraic equation. It will be described as follows.

TRANSFER MATRIX METHOD (TMM) DERIVATION FOR TE MODE

We consider two horizontally connected regions with reflective indices of n_c and n_v, with $x=X$ in their interface, so the solution of Eq. 5 in these two regions is:

$$\phi_c = A_c e^{-i\gamma_c x} + B_s e^{i\gamma_c x} \qquad x < X$$

$$\phi_s = A_s e^{-i\gamma_s x} + B_s e^{i\gamma_s x} \qquad x > X$$

Where A and B are undefined amplitudes of forward and backward waves, respectively. In this case, we have

$C_s = Q_{c\to s}C_c$ with $C_c = [A_c \quad B_c]^T$, $C_s = [A_s \quad B_s]^T$, $C_s = Q_{c\to s}C_c$ and $Q_{c\to s}$ is the transfer matrix of coefficients of layer *'c'* to *'s'*. By imposing the boundary conditions of Eq. 4 in the above equations, we have:

$$Q_{c\to s} = \frac{1}{2}\frac{n_c}{n_s}\begin{bmatrix} (1+\frac{\gamma_c}{\gamma_s})e^{+i(\gamma_s-\gamma_c)X} & (1-\frac{\gamma_c}{\gamma_2})e^{+i(\gamma_s+\gamma_c)X} \\ (1-\frac{\gamma_c}{\gamma_s})e^{-i(\gamma_s+\gamma_c)X} & (1+\frac{\gamma_c}{\gamma_s})e^{-i(\gamma_s-\gamma_c)X} \end{bmatrix} \quad (8)$$

A similar complete analytical solution is presented in (Jiani, Khorasani, Rashidian, & Mohammdi, 2009). Considering the reported results and according to Eq. 8, we can conclude that, by inserting a layer with a constant index of n0 between 'c' and 's' layers, its transfer matrix must be equal to the identity matrix, because their indices are constant in a homogenous medium.

$$Q_{0^+\to d^-} = I \quad (9)$$

Also, the solution of Eq. 5 in this medium when $n=n_0$ is:

$$\phi(x) = Ae^{-i\gamma_0 x} + Be^{i\gamma_0 x}, \quad 0 < x < d \quad (10)$$

Where $\gamma_0 \equiv \sqrt{k_0^{\,2}n_0^2 - \beta^2}$. By substituting Eq. 5 in Eq. 6 and Eq. 7, a homogenous set of equations generates, and by imposing the homogeneity condition, the dispersion relation for this kind of waveguide is obtained easily as:

$$e^{2id\gamma_0} = \frac{\gamma_0+\gamma_c}{\gamma_0-\gamma_c}\cdot\frac{\gamma_0+\gamma_s}{\gamma_0-\gamma_s} \quad (11)$$

DTMM DERIVATION AND DISPERSION RELATION FOR TE MODE

Consider the case that the reflective index of the middle layer varies as a continuous function. For this situation, the related transfer matrix is not the identity matrix any longer and solution of Eq. 7 is:

$$\phi(x) = A(x)e^{-i\gamma_0(x)\,x} + B(x)e^{i\gamma_0(x)x}, \quad 0 < x < d \quad (12)$$

Calculating the transfer matrix of the middle layer equals to finding a relation between the coefficients of this layer at two different *x*-points. Then, it is possible to remove the dependence of the coefficients to *x* and reduce the complication of the relation.

It is known that:

$$C(x+\Delta x) = Q_{x\to x+\Delta x}C(x) \quad (13)$$

Where $C_x \equiv [A(x) \quad B(x)]^T$. And we have:

$$\Delta C(x) \approx C(x+\Delta x) - C(x) \quad (14)$$

By substituting Eq. 13 in Eq. 14, we have:

$$\frac{\Delta C(x)}{\Delta x} = \frac{Q_{x\to x+\Delta x} - I)C(x)}{\Delta x} \quad (15)$$

Where **I** is defined as the identity matrix. If $\Delta x \to 0$:

$$dc(x) = \{H(x)\}c(x)dx \quad (16)$$

Or

$$C'(x) = [H(x)]\{C(x)\} \quad (17)$$

And the kernel matrix of *H(x)* is defined as:

$$H(x) = \begin{bmatrix} H_{11}(x) & H_{12}(x) \\ H_{21(x)} & H_{22}(x) \end{bmatrix} = \lim_{\Delta x\to 0} \frac{1}{\Delta x}(Q_{c\to s} - I)$$

To obtain the kernel matrix define some substitutions as follows (Jiani, Khorasani, Rashidian, & Mohammdi, 2009):

$$\begin{matrix} K_1 \equiv K & K_2 \equiv K + K_\gamma \Delta k & K_x \equiv \frac{\partial K}{\partial x} \\ K_1' \equiv K_x & K_2' \equiv K_x + K_{x\gamma}\Delta k & K_{xk} \equiv \frac{\partial^2 K}{\partial x \partial \gamma} \end{matrix} \quad (18)$$

In which K is substituted by E and H when $E = e^{-ix\gamma(x)}$, $H = e^{+ix\gamma(x)}$. After some algebraic simplifications, the kernel matrix would be:

$$H(x) = \frac{\gamma_0'(x)}{2\gamma(x)}\begin{bmatrix} -1+2i\gamma_0(x)x & e^{2i\gamma_0(x)x} \\ e^{-2i\gamma_0(x)x} & -1-2i\gamma_0(x)x \end{bmatrix}$$

Or

$$H(x) = i\gamma_0'(x)x\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \frac{\gamma_0'(x)}{2\gamma(x)}\begin{bmatrix} 1 & -e^{2i\gamma_0(x)x} \\ -e^{-2i\gamma_0(x)x} & 1 \end{bmatrix} \quad (19)$$

The first part of Eq. 19 shows the phase changes and the second part shows the interaction between forward and backward waves. So for a gradual variation of the refractive index, we can neglect the second part which can reduce the complication. Regarding to perturbation theory we can have (Jiani, Khorasani, Rashidian, & Mohammdi, 2009):

:

$$\{C(x_2)\} = \exp\left[\int_{x_1}^{x_2} H(x)dx\right]\{C(x_1)\} \quad (20)$$

Where $\exp(M) = I + \sum_{n=1}^{\infty}\frac{1}{n!}M^n$. Eliminating the second part of *H(x)*, we have:

$$\{C(d_-)\} = \exp\left[\int_0^d H(x)dx\right]\{C(0_+)\} = \exp(M)\{C(0_-)\} \quad (21)$$

So

$$Q_{0^+\to d^-} = \exp\left[\int_0^d H(x)dx\right] \quad (22)$$

Where:

$$H(x) = i\gamma_0'(x)x\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (23)$$

Now, obtaining the relation between coefficients for matrix of Eq. 22 and imposing the boundary conditions in Eq. 4, the following dispersion relation can be found:

$$e^{2i\int_0^d \gamma_0(t)dt} = \frac{\gamma_0(0)+\gamma_c}{\gamma_0(0)-\gamma_c} \cdot \frac{\gamma_0(d)+\gamma_s}{\gamma_0(d)-\gamma_s} \quad (24)$$

SOLVING THE DISPERSION RELATION; NEWTON- RAPHSON METHOD

We use Newton- Raphson method to obtain the roots of dispersion relation Eq. 23. To have the initial guess in this method, we use the results of Lu and Zhu paper that uses WKB method for this kind of waveguide (Zhu, Chen, & Tang, Leaky modes of optical waveguides with varied refractive index for microchip optical interconnect applications— Asymptotic solutions, 2008). Then, we define initial guess for TE mode as:

$$\sqrt{k_0^2 n_0^2(\xi) - \beta_m^2} = \frac{2i}{d} LambertW(P, \frac{-i^{-q}d}{4}(H_1 H_2)^{1/4}) \quad (25)$$

With the definitions of:

$H_1 = k_0^2(n_0^2(\xi) - n_1^2)$, $H_2 = k_0^2(n_0^2(\xi) - n_2^2)$, $p = -1, -2, \ldots,$ $q = 1,2,3,$ and $m = -4(p+1) + q + 1$

Considering:

$$F(\beta) = \frac{\gamma_0(0,\beta)+\gamma_c(\beta)}{\gamma_0(0,\beta)-\gamma_c(\beta)} \cdot \frac{\gamma_0(d,\beta)+\gamma_s(\beta)}{\gamma_0(d,\beta)-\gamma_s(\beta)} - e^{2i\int_0^d \gamma_0(t,\beta)dt} = 0 \quad (26)$$

and substituting it in the following Newton-Raphson relation, we can compute the propagation constant for each mode.

$$\beta_m^{k+1} = \beta_m^k - \frac{F(\beta_m^k)}{F(\beta_m'^k)} \quad (27)$$

NUMERICAL RESULTS

At first, we chose an asymmetric waveguide when nc ≠ ns. The dispersion relation of TE mode must be used to compute the leaky modes, when the reflective index is varied gradually.Assume that for example nc =2.10, ns =3.17, d=2 μm and the index profile is::

$$n_0(x) = 3.3\left[1 - 0.01\left(\frac{x-1.0}{2.5}\right)^2\right]$$

The free space wavelength is λ =1.55 μm . In these calculations, after a few iterations the mode propagation constant can be found for each mode. The results are shown in Fig. 2. As seen, the solution of Eq. 25 is not close to the exact solution, and more steps of iterations are needed. Based on the derivation of differential matrices and the transfer relation, the exact solutions (Zhu, Chen, & Tang, Leaky modes of optical waveguides with varied refractive index for microchip optical interconnect applications— Asymptotic solutions, 2008), i.e. leaky modes, can be found. The results are shown in Table 1.

Table 1. Solution of the first example

Mode(m)	$\beta(DTMM)$	$\beta(EXACT)$	Error
8	5.7952+ 2.7464i	3.720599+5.3860768i	0.5128
9	3.7155+ 5.4103i	2.903792+8.3614252i	0.4663
10	2.9051+ 8.3849i	2.617303+10.934894i	0.2891
11	2.6208+10.9579i	2.498615+13.234072i	0.2023
12	2.5032+13.2576i	2.448795+15.362522i	0.1560
13	2.4543+15.3872i	2.432718+17.377481i	0.1277

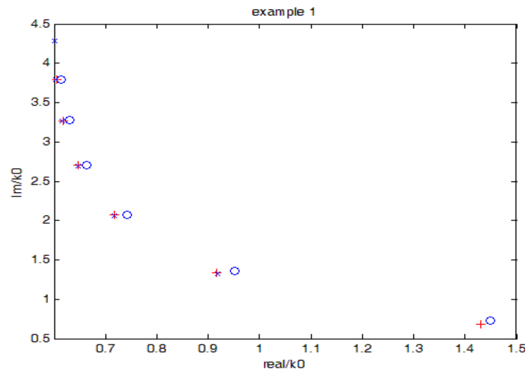
As the second example, a sinusoidal index profile was studied with the following description:

$$n_0(x) = 3.3 + .15\sin(2\pi x)$$

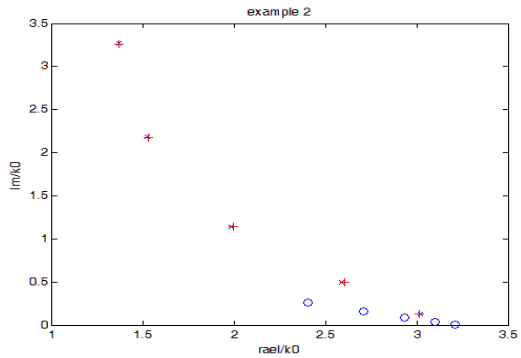
Together with nc =3.10, ns =3.00 and d=1 μm . The free-space wavelength is still λ =1.55 μm . The results of dispersion relations are excellent.

Mode(m)	$\beta(DTMM)$	$\beta(EXACT)$	**Error**
2	12.22180+0.52154i	12.2277+ 0.53368i	0.0082
3	10.55669+2.00745i	10.47991+1.99504i	0.0073
4	8.08928 +4.60537i	8.02948+ 4.60537i	0.0071
5	6.19182 +8.83170i	6.16103+ 8.87021i	0.0046
6	5.54474+13.21640i	5.53044+13.24802i	0.0024

Table 2. Solution of the second example



(a)



(b)

Fig 2:(a) Solution for example 1, and (b) example 2. The asymptotic solutions (WKB) are marked by "○," DTMM results are denoted by "+," and the exact solutions are shown by "□ ".

CONCLUSIONS

A new analytical solution of the wave equation, considering the leaky modes, was presented. The propagation constant, which has been obtained by DTMM, depends on the asymmetric non-magnetic waveguide structure. We concluded that when the reflective index changes gradually, the related equations for TE mode can be simplified. It was shown that, the proposed method is much closer to exact solution than WKB method and the performance is very proper when the index of modes is large.

REFRENCES

Botez, D., Mawst, L., & Peterson, G. (1988). Resonant leaky-wave coupling in linear arrays of antiguides. *Electron. Len*, 1328-1330.

Hu, J., & Menyuk, C. (2009). Understanding leaky modes Slab waveguide revisited. *Adv.Opt. Photon*, 1.58-106.

Jiani, M., Khorasani, S., Rashidian, B., & Mohammdi, S. (2009). Formulation of differential transfer matrix method in cylindrical geometry.*Proc of SPIE*, 80.7597-75971.

Khorasani, S., & Mehrany, K. (2003). Differential transfer-matrix method for solution of one-dimensional linear nonhomogeneous optical structures. *J. Opt. Soc. Amer*, 20.91-96.

Lee, J., Chung, Y., Coldren, L., & Dagli, N. (1995). On leaky mode approximations for modal expansion in multilayer open waveguides. *IEEE J. Quantum Electron*, 31.1790-1802.

Stetten, H., & Kneubuhl, F. (1968). Dielectric tube resonators for infrared and submillimiter wave lasers. *Phys.Len*, 612.

Stowell, D., &Tausch,J.(2009). Variational formulation for guided and leaky modes in multilayer dielectric waveguides. *Commun. Comput. Phys*, 564-579.

Thayagarajan, K., Shenoy, M., & Ghatak, A. (1978). Accurate numerical method for the calculation of bending loss in optical waveguides using a matrix approach. *Opt.Len*, 296.

Thyagarajan, K., Diggavi, S., & Ghatak, A. (1978). Analytical investigation of leaky and absorbing planar structureS. *Opt and Quantum Electron,* 131-137.

Ye, R., & Yevick, D. (2001). Noniterative calculation of complex propagation constants in planar waveguides. *J. Opt. Soc. Amer*, 2819-2822.

Zhu, Z., & Lu, Y. (2006). Leaky modes of slab waveguides—Asymptotic solutions. *J.Lightw. Technol*, 24.1619-1623.

1. Zhu, Z., Chen, Z., & Tang, S. (2008). Leaky modes of optical waveguides with varied refractive index for microchip optical interconnect applications— Asymptotic solutions. *Microelectron. Reliab*, 48.555-562