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Determination of Escape Speed from de Broglie-Bohm Interpretation

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ABSTRACT

In this study, we apply the standard quantization procedure to the Newtonian equation to obtain the Schrödinger equation. The wave function is obtained and subsequently the de Broglie-Bohm interpretation is applied to the wave function to yield the formulas for escape speed. It is shown that the usual Newtonian formula for escape speed is purely resulted from taking the asymptotic form of Bessel functions. We then extend our work to hydrogen atom and show that the work done to eject the electron away from proton is in discrete form. The ionization energy for ground state of hydrogen atom from escape kinetic energy method is obtained.

Key words: de Broglie-Bohm interpretation, escape speed, Newton's law of gravitation, ionization energy of hydrogen atom
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INTRODUCTION

If you toss an object vertically upwards, the object will reach its highest point and stop momentarily before falling down. However, this is not always the case. If the object has large enough speed, it can successfully move away from the Earth and never return. The minimum value of speed an object requires to escape from a gravitating body is called the escape speed. In classical mechanics, we obtain the escape speed of an object by applying the conservation of energy. The total energy for an object at the surface of any planet must be the same as in the outer space. According to Newton's law of gravitation, the gravitational potential energy associated with any two objects is given by

$$PE_g = -\frac{Gm_1m_2}{d} \tag{1}$$

G is the Newton's gravitational constant, m_1 and m_2 are masses of spherically symmetric objects O_1 and O_2 respectively, and d is the distance between the two objects. Object O_1 must have the kinetic energy KE_1 in order that it can escape away from object O_2 . The kinetic energy of the object O_1 is given by

$$KE_1 = \frac{1}{2}m_1v^2 \tag{2}$$

where v is the speed of object O_1. Thus, the total mechanical energy E of object O_1 which is escaping from object O_2 is given by the following equation (Vuille, Serway, & Faughn, 2009).

$$E = \frac{1}{2}m_1v^2 - \frac{Gm_1m_2}{d} \tag{3}$$

From the principle of conservation of energy, the total mechanical energy E of object O_1 is always a constant value. In addition, the value of E could be negative, positive or zero. If E is negative, the object O_1 will be returning back to object O_2. On the other hand, if E is positive or zero, the object O_1 will be escaping forever from object O_2. However in the case of zero E, object O_1 will escape away with an initial speed which is just large enough to make the speed of O_1 asymptotically toward zero when it reaches infinity. This initial speed is called the escape speed. We set $E = 0$ in equation (3) and solve for v to obtain the escape speed as

$$v_{esc} = \sqrt{\frac{2Gm_2}{d}} \tag{4}$$

People typically apply Newton's law of motion for studying the definite motion of a single macroscopic object. In this study, we employ a Schrodinger equation to describe an object O_1 which is moving in a gravitational field. It seems weird to use the Schrodinger equation for illustrating the motion of a single object which has a definite trajectory. However, the weirdness will be gone if we employ the de Broglie-Bohm interpretation (Atiq, Karamian, & Golshani, 2009) to interpret the wave function of Schrodinger equation rather than the usual probabilistic Copenhagen interpretation (Griffiths & Dick, 1995). This is due to the ability of de Broglie-Bohm interpretation to give the exact trajectory of a single particle. In the next section, we summarize the de Broglie-Bohm interpretation of wave function. In section 3, we write down the Schrodinger equation for an object which is moving with the escape speed in a gravitational field. We then solve the Schrodinger equation for the wave function and obtain the wave function in asymptotic form. Afterwards, we apply the de Broglie-Bohm interpretation to the asymptotic form of wave function to obtain the formulas for escape speed. In section 4, we extend the work to hydrogen atom where the ionization energy is found. Finally, we conclude in section 5.

DE BROGLIE-BOHM INTERPRETATION

Suppose a non-relativistic particle of mass m moves in space in which it is acted upon by a force $-\nabla V$. For simplicity, we confine the particle to move in only one spatial dimension. The energy equation for this particle is given by

$$E = \frac{p^2}{2m} + V \tag{5}$$

E, p and m are the total energy, momentum and mass of the particle respectively. In addition, $V = V(x,t)$ is the potential energy where x is position of the particle and t is time. We now employ the standard replacements to replace the total energy E and momentum p in equation (5) by the energy and momentum operator which act on the wave function $\Psi(x,t)$. The energy and momentum operators are given as

$$\hat{E} = i\hbar\frac{\partial}{\partial t} \tag{6}$$

and

$$\hat{p} = -i\hbar\frac{\partial}{\partial x} \tag{7}$$

respectively, where $i = \sqrt{-1}$ is a complex number and $\hbar$ is the reduced Planck's constant. Hence equation (5) would become

$$i\hbar\frac{\partial\Psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial^2 x} + V(x,t)\Psi(x,t) \tag{8}$$

Equation (8) is known as the Schrodinger equation, where we may also write it as follows

$$i\hbar\frac{\partial\Psi(x,t)}{\partial t} = \hat{H}\Psi(x,t) \tag{9}$$

$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial^2 x} + V(x,t)$ is called the Hamiltonian operator. The wave function $\Psi(x,t)$ can also be written as follows.

$$\Psi(x,t) = R(x,t)\exp\left[\frac{iS(x,t)}{\hbar}\right] \tag{10}$$

R and S here are real functions. Substituting the wave function $\Psi(x,t)$ (10) into (8) and after taking the derivatives, we obtain the following two equations from the real and imaginary parts (Holland, 1995).

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + V + Q = 0 \tag{11}$$

and

$$\frac{\partial R^2}{\partial t} + \nabla\cdot\left(R^2\frac{\nabla S}{m}\right) = 0 \tag{12}$$

where $Q = -\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}$. Equation (12) is a continuity equation for probability density $R^2\,(=|\Psi|^2)$ as viewed from the Copenhagen interpretation. However, we can regard equation (11) as a modified Hamilton-Jacobi equation in which an extra potential Q is added to the usual Hamilton-Jacobi equation. Therefore, the particle now is not only acted upon by a classical potential but also by an extra potential. Hence we could now identify momentum p of a particle as

$$p = \frac{dS}{dx} \tag{13}$$

Equation (13) is called guidance equation where

$$p = \frac{\partial L}{\partial \dot{x}} \tag{14}$$

Here L is the Lagrangian while dot notation denotes differentiation with respect to time. This equation plays an important role of giving the speed, $\dot{x}$ and trajectory $x(t)$ of the particle. In fact, the extra potential Q in equation (11) is responsible for all the quantum effects and being named as quantum potential. Thus we expect that the quantum potential Q will be sufficiently small and negligible in classical regime of physics.

ESCAPE SPEED FOR GRAVITATING OBJECTS

We start by employing a spherical coordinate system to specify the positions of objects O_1 and O_2. However for simplicity, the origin of coordinate system is chosen to coincide with the centre of object O_2 and object O_1 is allowed to travel only along the line of radial coordinate.

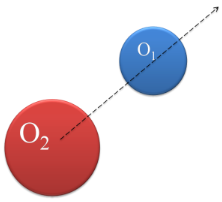

Fig.1: Object O_1 travels along a line of radial coordinate emanating from the centre of object O_2.

The total energy of an object O_1 which is escaping from another object O_2 is given as follows

$$E = \frac{1}{2}m_1\dot{r}^2 - \frac{Gm_1m_2}{r} \tag{15}$$

$\dot{r}$ is the speed of object O_1 travelling along the line of radial coordinate. In the study, equation (15) also is the Hamiltonian for object O_1, hence we write the Lagrangian L for the object O_1 as follows

$$L = \frac{1}{2}m_1\dot{r}^2 + \frac{Gm_1m_2}{r} \tag{16}$$

Immediately, we obtain the momentum of object O_1 as follows

$$p_{1r} = \frac{dL}{d\dot{r}} = m_1\dot{r} \tag{17}$$

Consequently, the guidance equation (13) becomes

$$m_1\dot{r} = \frac{dS}{dr} \tag{18}$$

Setting the total energy $E = 0$ and rewriting the kinetic energy of O_1 in term of momentum Π_{1r}, equation (15) becomes

$$\frac{{\Pi_{1r}}^2}{2m_1} - \frac{Gm_1m_2}{r} = 0 \tag{19}$$

where $\Pi_{1r} = m_1\dot{r}_{esc(1)}$ in which $\dot{r}_{esc(1)}$ here denotes the escape speed of object O_1. We proceed to replace momentum Π_{1r} by an operator $-i\hbar\,\partial/\partial r$ to obtain the following Schrodinger equation

$$\frac{\hbar^2}{2m_1}\frac{d^2\psi}{dr^2}+\frac{Gm_1m_2\psi}{r}=0 \tag{20}$$

ψ is the wave function of object O_1. Solving equation (20) gives the wave function ψ as follows (Polyanin & Zaitsev, 2002).

$$\psi=\sqrt{r}\left[C_1J_1\left(\sqrt{\frac{8Gm_1{}^2m_2r}{\hbar^2}}\right)+C_2Y_1\left(\sqrt{\frac{8Gm_1{}^2m_2r}{\hbar^2}}\right)\right] \tag{21}$$

where $J_1\left(\sqrt{\frac{8Gm_1{}^2m_2r}{\hbar^2}}\right)$ and $Y_1\left(\sqrt{\frac{8Gm_1{}^2m_2r}{\hbar^2}}\right)$ are Bessel functions of first and second kind respectively, of order 1, while C_1 and C_2 are arbitrary constants. Interestingly, we study two special conditions

$$\sqrt{\frac{8Gm_1{}^2m_2r}{\hbar^2}}\gg 1 \tag{22}$$

and

$$\sqrt{\frac{8Gm_1{}^2m_2r}{\hbar^2}}\ll 1. \tag{23}$$

To have a real physical situation, the arguments of Bessel functions cannot equal to zero. These conditions (22) and (23) are imposed as in (Lea, 2004) for obtaining the asymptotic form of Bessel functions. For these two cases, we set $C_1=A$ and $C_2=iA$, where A is a positive constant.

Case 1 , condition: $\sqrt{\frac{8Gm_1{}^2m_2r}{\hbar^2}}\gg 1$. The wave function (21) reduces to

$$\psi=A\left(\sqrt{\frac{r\hbar^2}{2\pi^2Gm_1{}^2m_2}}\right)^{\frac{1}{2}}\exp\left[i\left(\sqrt{\frac{8Gm_1{}^2m_2r}{\hbar^2}}-\frac{3\pi}{4}\right)\right] \tag{24}$$

We then obtain the phase $\frac{S}{\hbar}$ of the above wave function (24) and write it as follows

$$\frac{S}{\hbar}=\sqrt{\frac{8Gm_1{}^2m_2r}{\hbar^2}}-\frac{3\pi}{4} \tag{25}$$

We proceed to compute $\frac{dS}{dr}$ and substitute it into guidance equation (18), we obtain an equation as follows

$$m_1\dot{r}_{esc(1)}=\sqrt{\frac{2Gm_1{}^2m_2}{r}} \tag{26}$$

Finally, solving above equation (26) for escape speed $\dot{r}_{esc(1)}$, we obtain

$$\dot{r}_{esc(1)}=\sqrt{\frac{2Gm_2}{r}} \tag{27}$$

This is indeed the usual Newtonian formula for escape speed.

Case 2 , condition: $\sqrt{\frac{8Gm_1{}^2m_2r}{\hbar^2}}\ll 1$. The wave function (21) reduces to

$$\psi=Ar\sqrt{\frac{2Gm_1{}^2m_2}{\hbar^2}}-i\frac{A}{\pi}\sqrt{\frac{\hbar^2}{2Gm_1{}^2m_2}} \tag{28}$$

We then compute the phase $\frac{S}{\hbar}$ of the above wave function (28) and write it as follows

$$\frac{S}{\hbar}=-\tan^{-1}\left[\frac{\hbar^2}{2G\pi m_1{}^2m_2r}\right] \tag{29}$$

We proceed to compute $\frac{dS}{dr}$. Substituting it into guidance equation (18), we obtain an equation as follows

$$m_1\dot{r}_{esc(1)}=\frac{(\hbar^3/2\pi Gm_1{}^2m_2)}{r^2[1+(\hbar^2/2\pi Gm_1{}^2m_2r)^2]} \tag{30}$$

Equation (30) seems to be complicated; however we can reduce it to a simpler form by considering the following approximation: $1+(\hbar^2/2\pi Gm_1{}^2m_2r)^2\approx(\hbar^2/2\pi Gm_1{}^2m_2r)^2$. The approximation is convincing if the condition (23) is valid. Equation (30) thus reduces to

$$m_1\dot{r}_{esc(1)}=\frac{(\hbar^3/2\pi Gm_1{}^2m_2)}{r^2(\hbar^2/2\pi Gm_1{}^2m_2r)^2} \tag{31}$$

At last, solving above equation (31) for escape speed $\dot{r}_{esc(1)}$, we get

$$\dot{r}_{esc(1)}=\frac{2\pi Gm_1m_2}{\hbar} \tag{32}$$

Escape speed (32) has the reduced Planck constant $\hbar$, besides this, it is independent on the distance r. However we have to keep in mind that equation (32) can only be applied if the condition (23) is satisfied.

ESCAPE SPEED FOR ELECTRON IN HYDROGEN ATOM

The hydrogen atom consists of a proton of charge q and an electron of charge $-q$ in which the electric potential energy associated with the pair of proton and electron is given by

$$PE_e=-\frac{k_eq^2}{d} \tag{33}$$

k_e is the Coulomb's constant. In addition, their gravitational potential energy is as follows

$$PE_g=-\frac{Gm_em_p}{d} \tag{34}$$

m_e and m_p are masses of electron and proton respectively. Based on the physical values of constants m_e, m_p, k_e, q and G, we get $PE_e\gg PE_g$. Therefore we neglect the gravitational potential energy. We then employ a spherical coordinate system to indicate the position of proton and electron where the origin of coordinate system is chosen to coincide with the centre of proton. The total energy for an electron which is escaping from the proton along the line of radial coordinate is given by

$$E=\frac{1}{2}m_e\dot{r}^2-\frac{k_eq^2}{r} \tag{35}$$

and the Lagrangian for electron is

$$L=\frac{1}{2}m_e\dot{r}^2+\frac{k_eq^2}{r} \tag{36}$$

We take an assumption that the electron is ejected along the line of radial coordinate. Setting total energy (35) zero and rewriting the kinetic energy in term of momentum Π_{er}, equation (35) becomes

$$\frac{\Pi_{er}{}^2}{2m_e}-\frac{k_eq^2}{r}=0 \tag{37}$$

where $\Pi_{er}=m_e\dot{r}_{esc(e)}$ in which $\dot{r}_{esc(e)}$ denotes the escape speed of electron. Then we write down the Schrodinger equation as follows

$$\frac{\hbar^2}{2m_e}\frac{d^2\psi}{dr^2}+\frac{k_e q^2\psi}{r}=0 \quad (38)$$

The wave function ψ of (38) is given by (Polyanin & Zaitsev, 2002).

$$\psi=\sqrt{r}\left[C_1 J_1\left(\sqrt{\frac{8m_e k_e q^2 r}{\hbar^2}}\right)+C_2 Y_1\left(\sqrt{\frac{8m_e k_e q^2 r}{\hbar^2}}\right)\right] \quad (39)$$

Similar in the previous section, let us now take $C_1=\alpha$ and $C_2=i\alpha$, where α is a positive constant. Besides this, we take the assumption of $\sqrt{\frac{8m_e k_e q^2 r}{\hbar^2}} \ll 1$. Using the asymptotic form of Bessel functions, the wave function (39) becomes

$$\psi=\alpha r\sqrt{2m_e k_e q^2/\hbar^2}-i\frac{\alpha}{\pi\sqrt{2m_e k_e q^2/\hbar^2}} \quad (40)$$

The phase $\frac{S}{\hbar}$ of the above wave function is calculated and written as follows

$$\frac{S}{\hbar}=-\tan^{-1}\left(\frac{\hbar^2}{2\pi m_e k_e q^2 r}\right) \quad (41)$$

As in the last section, $\frac{dS}{dr}$ is calculated. We thus have a following equation:

$$m_e\dot{r}_{esc(e)}=\frac{(\hbar^3/2\pi k_e q^2 m_e)}{r^2[1+(\hbar^2/2\pi r k_e q^2 m_e)^2]} \quad (42)$$

If $\sqrt{\frac{8m_e k_e q^2 r}{\hbar^2}} \ll 1$, we have $1+(\hbar^2/2\pi r k_e q^2 m_e)^2 \approx (\hbar^2/2\pi r k_e q^2 m_e)^2$. Applying this approximation to equation (42), the escape speed of electron in hydrogen atom becomes

$$\dot{r}_{esc(e)}=\frac{2\pi k_e q^2}{\hbar} \quad (43)$$

The kinetic energy of electron when electron starts to escape away from the proton is given by

$$KE_{i(e)}=\frac{m_e}{2}\left(\frac{2\pi k_e q^2}{\hbar}\right)^2 \quad (44)$$

We obtain the ionization energy (the work done of moving the electron away from the proton) as follows

$$E_{ion(e)}=-\left(KE_{f(e)}-KE_{i(e)}\right) \quad (45)$$

$KE_{f(e)}$ (final kinetic energy of electron) is equal to zero when the electron reaches infinity while $KE_{i(e)}$ (initial kinetic energy of electron) is obtained from equation (44). The ionization energy (45) becomes

$$E_{ion(e)}=\frac{4\pi^2 {k_e}^2 q^4 m_e}{2\hbar^2} \quad (46)$$

However, we can write $4\pi^2$ as a following series (Spiegel & Liu, 1999).

$$\frac{24}{1^2}+\frac{24}{2^2}+\frac{24}{3^2}+\frac{24}{4^2}+\ldots\ldots=4\pi^2 \quad (47)$$

Consequently, equation (46) turns out to be

$$E_{ion(e)}=24\left[\frac{{k_e}^2 q^4 m_e}{2\hbar^2}\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\ldots\right)\right] \quad (48)$$

The first term in the square bracket is the ionization energy for ground state of hydrogen atom as from the Bohr model. However the ionization energy (48) is appearing in the form of summation of many discrete terms.

DISCUSSIONS AND CONCLUSIONS

We have shown that the usual Newtonian formula for escape speed is purely resulted from taking the asymptotic form of Bessel functions. We use formula (27) to calculate escape speed of objects for whatever conditions. However in this paper, we show that this is simply not true. For instance when the condition (23) is obeyed, we need to employ equation (32) rather than the usual Newtonian formula (27) to get the escape speed. In this paper, we only consider two special cases where the objects' masses and separation of the objects satisfy one of the two conditions (22) or (23). To be more general, we should consider the whole Bessel functions and not only the asymptotic form of Bessel functions. We then extend the work to hydrogen atom. We study the escape speed of electron and ionization energy of hydrogen atom. The asymptotic form of the wave function is covered for a small range of value of r (few energy levels) and not only for the Bohr radius (lowest energy level). Besides this, the energy of electron in hydrogen atom is in the discrete form (quantized). We thus find that ionization energy (48) is consisting of sum of many discrete terms. If we apply the same formalism as done in section 4 to section 3.2, we are led to conclude that the energy of object O_1 is also quantized. The energy for ground state is $E_{esc(1)}=\frac{G^2{m_1}^3{m_2}^2}{2\hbar^2}$. For a system consisting of two gravitating object that satisfies the condition (23), the initial kinetic energy required by an object to escape infinitely far from another is in discrete form.

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