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Generalized symmetries and higher-order Conservation laws of the Camassa–Holm equation

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ABSTRACT

In the present paper, we derive generalized symmetries of order three of the Camassa–Holm equation by infinite prolongation of a generalized vector field and applying infinitesimal symmetry criterion. In addition, one-dimensional optimal system of Lie subalgebras investigated by applying the adjoint representation. Furthermore, determining equation for multipliers and the 2-dimensional homotopy formula employed to construct higher–order conservation laws for the Camassa–Holm equation.

Keywords: Camassa–Holm equation, Generalized Symmetry, Optimal system, Conservation

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INTRODUCTION

In this paper, we consider the Camassa–Holm equation

$$u_t - u_{tx^2} + ku_x + 3uu_x = 2u_x u_{x^2} + uu_{x^3}, \qquad k \in R \tag{1}$$

This equation introduced first as a model describing propagation of unidirectional gravitational waves in shallow water approximation, with u representing the fluid velocity at time t in the x direction, (Camassa & Holm, 1993; Camassa, Holm, & Hyman, 1994; Constantin & Escher, 1998) or equivalently, the high of water's free surface above a flat bottom. The equation (1) has a bi-Hamiltonian structure(Fuchssteiner & Fokas, 1981), and is completely integrable (Constantin, 2001). Moreover, the Camassa–Holm equation is a re-expression of geodesic flow on the diffeomorphism group of the line. Holm, Marsden, and Ratiu (1998) have, shown that Camassa–Holm equation in n dimensions describes geodesic motion on the diffeomorphism group of R^n with respect to metric given by the H^1 norm or Euclidean Fluid velocity. Misiołek (1998) has shown that Camassa–Holm equation represents a geodesic flow on the Bott–Virasoro group. Kouranbaeva (1999) has shown that equation (1) (for the case $k = 0$) is a geodesic spray of the weak Riemannian metric on the diffeomorphism group of the line or the circle obtained by the right translation of the H^1inner product over the entire group. Generalized symmetries first make their appearance the original paper of E. Noether on the correspondence between symmetries of variational problems and conservation laws of the associated Euler-Lagrange equations. The terminology generalized refers to the fact that the infinitesimal generators allowed to depend on derivatives of the dependent variables, which makes the corresponding group transformations nonlocal. More recently, these "generalized symmetries" have proved to be of importance in the study of nonlinear waves equations, where it appears that the possession of an infinite number of such symmetries, is a characterizing property of "solvable " equations. The symmetry group for the Camassa–Holm equation was first derived by Clarkson, Mansfield, and Priestley (1997). Actually, Clarkson, Mansfield and Priestley found the symmetry group related to a large class of partial differential equations that also contains the Camassa–Holm equation. The particular case $k = 0$ has been studied by N. Bila and C. Udriste in (Bila, 1999).

In this paper, we derive generalized symmetries of order three for the Camassa–Holm equation by using evolutionary representative of a generalized vector field and then we obtain one–dimensional optimal system of Lie subagebras.

In continuation, we discuss on the conservation laws of the Camassa–Holm equation. It' well known that, the roll of multipliers thoroughly investigated, particularly in the construction of new and higher–order conservation laws. In fact, in the latter case, there is a one to one correspondence between multipliers and conservation laws due to a homotopy integral formula that a knowledge of a multiplier, by formula leads to a conserved flow (Hereman, 2006). R. Naz, I. Naeem and S. Abelman found conservation laws of the Camassa-Holm equation of first order by using the variational derivative approach (Naz, Naeem, & Abelman, 2009). The second order conservation laws for the Camassa–Holm equation with k=0 have been derived in a recent paper by A.H. Kara and A.H. Bokhari by a non variational approach (Kara & Bokhari, 2011). In this paper, we construct higher–order conservation laws of the Camassa-Holm equation.

GENERALIZED SYMMETRIES

Consider a system of n-th order differential equations in p independent and q dependent variables as follows

$$\Delta_v\left(x, u^{(n)}\right) = 0\,, \quad v = 1, \dots, N \tag{2}$$

Involving $x = (x^1, \dots, x^p)$, $u = (u^1, \dots, u^q)$ and the derivatives of u with respect to x up to order n. A generalized vector field will be a expression of the form

$$v = \sum_{i=1}^{p} \xi^i[u] \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{q} \phi_\alpha\,[u] \frac{\partial}{\partial u^\alpha}\ (3) \tag{3}$$

in which coefficient functions ξ^i, ϕ_αdepend on x, u and derivatives of u. By the prolongation formula of theorem (2.36) in (Olver, 2000), we can define the prolonged generalized vector field

$$pr^{(n)}v = v + \sum_{\alpha=1}^{q} \sum_{\#J \le n} \phi_\alpha^j[u] \frac{\delta}{\delta u_J^{\alpha,}} \tag{4}$$

whose Coefficients are determined by the formula

$$\phi_\alpha^J = D_J\left(\phi_\alpha - \sum_{i=1}^{p} \xi^i u_i^\alpha\right) + \sum_{i=1}^{p} \xi^i\, u_{J,i}^\alpha \tag{5}$$

Since all the prolongation of v have, the same general expression for their coefficient functions ϕ_{α}^J, it is helpful to pass to the infinite prolongation, and take care of all the derivatives at once. Specially, given a generalized vector field v, its infinite prolongation is the formally infinite sum

$$prv = \sum_{i=1}^{p} \xi^i \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_J \phi_\alpha^J \frac{\partial}{\partial u_J^\alpha}, \tag{6}$$

where each ϕ_α^J has given by (5), and the sum in (6) now extends over all multi indices $j = (j_1, \dots, j_k) for \;\; k \ge 0, 1 \le j_k \le p$.

By the infinitesimal symmetry criterion in theorem (2.72) of (Olver, 2000), we can state the following result. A generalized vector field v is a generalized infinitesimal symmetry of a system of differential equations (2) if and only if

$$prv[\Delta_v] = 0, \quad v = 1, \dots, i, \tag{7}$$

For every smooth solution $u = f(x)$. Among all the generalized vector fields, those in which the coefficients $\xi^i[u]$ of the $\frac{\partial}{\partial x^i}$ are zero play a distinguished role. Let $Q[u] = \left(Q_1[u], \dots, Q_q[u]\right) \in \mathcal{A}^q$ be a q-tuple of differential functions. The generalized vector field

$$v_Q = \sum_\alpha^q Q_\alpha[u] \frac{\delta}{\delta u^\alpha} \tag{8}$$

is called an evolutionary vector field, and Q is called its characteristic. Note that according to (5), the prolongation of an evolutionary vector field takes a particularly simple form:

$$prv_Q = \sum_{\alpha,J} D_J Q_\alpha \frac{\partial}{\partial u_J^\alpha} \tag{9}$$

Any generalized vector field v as in (3) has an associated evolutionary representative v_Q in which the characteristic Q has entries

$$Q_\alpha = \phi_\alpha - \sum_{i=1}^{p} \xi^i u_i^\alpha, \quad \alpha = 1, \dots, q, \tag{10}$$

where $u_i^\alpha = \frac{\partial u^\alpha}{\partial x^i}$. These two generalized vector fields determine essentially the same symmetry. In principle, the computation of generalized symmetries of a given system of differential equations proceeds in the same way as the earlier computations of geometrical symmetries, but with the following added features:

First we should put the symmetry in evolutionary form v_Q. This has the effect of reducing the number of unknown functions from $p + q$ to just q, while simultaneously simplifying the computation of the prolongation prv_Q. One must then a priori fix the order of derivatives on which the characteristic $Q(x, u^{(m)})$ may depend. Therefore, by taking m not too large will yield important information on the general form of the symmetries. Finally, one must deal with the occurrence of trivial symmetries; the easiest way to handle these is to eliminate any superfluous derivatives in Q by substitution using the prolongation of the system. Suppose $v_Q = Q[u]\partial_u$ is a generalized symmetry in evolutionary form. Note that we can replace some derivatives of u occurring in Q by their corresponding expressions without changing the equivalence class of v. For instance, u_{xxt} replaced by $u_t + ku_x + 3uu_x - 2u_x u_{xx} - uu_{xxx}$ and so on. Thus, every symmetry is uniquely equivalent to one with characteristic

$$Q = Q(x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, u_{xxx}, u_{xtt}, u_{ttt}, \dots) \quad (11)$$

The prolongation of v_Q is given by

$$prv_Q = Q\partial_u + D_x Q\partial_{u_x} + D_t Q\partial_{u_t} + D_x^2 Q\partial_{u_{xx}} + D_x D_t Q\partial_{u_{xt}} + D_t^2 Q\partial_{u_{tt}} + \cdots \quad (12)$$

The infinitesimal condition (7) for invariance is then

$$D_t Q - D_x^2 D_t Q + kD_x Q + 3(u_x Q + uD_Q)$$
$$= 2(u_{xx} D_x Q + u_x D_x^2 Q) + u_{xxx} Q + uD_x^3 Q \quad (13)$$

which must be satisfied for all solutions. To calculate third order symmetries, we require

$$Q = Q(x, t, u, u_x, u_t, u_{xx} u_{xt}, u_{tt}, u_{xxx}, u_{xtt}, u_{ttt}) \quad (14)$$

So, upon substituting for $u_{xxt}, u_{xxxt}, u_{xxtt}, \dots$ in (13) according to the equation and after eliminating any dependence among the derivatives of the function of u, we are left to a complete system of determining PDEs. Therefore, the most general third–order characteristic function Q is

$$Q = \left(C_1 t - \frac{3}{2} C_2 u^2 + \frac{1}{2}(2u_{xx} - 2k)C_2 - 2C_3\right) u + C_3 u_{xx} + \frac{1}{2} u_x^2 + C_4) u_t + u_{xt^2} C_3 - \frac{1}{3} C_2 u_{ttt} - \frac{kt}{2} C_1 u_x + (-\frac{1}{3} C_2 u^3 - u^2 C_3 + \frac{C_5}{(k + 2u - 2_{u_{xx}})^{\frac{3}{2}}}) u_{xxx} - \frac{1}{2} C_3 u_x^3 + \frac{1}{2} C_1 k + C_1 u + \frac{1}{2} u_x (\frac{2C_5}{k + 2u - 2u_{xx})^{\frac{3}{2}}} + \frac{8}{3} C_2 u^3 + ((-2u_{xx} + k)C_2 + 9C_3) u^2 + 4\left(k - \frac{3}{2} u_{xx}\right) C_3 u + 2C_6 \quad (15)$$

where $C_1, \dots C_6$ are arbitrary constants. Therefore, we can state the following results:

Theorem 1. The most general third-order infinitesimal generalized symmetries of the Camassa-Holm equation is a R–linear combination of following six vector fields

$$Q_1 = u_x \quad , \quad Q_2 = u_t$$
$$Q_3 = -\frac{1}{2} ktu_x + tu_t + u + \frac{1}{2} k$$
$$Q_4 = \left(-\frac{3}{2} u^2 + \frac{1}{2}(2u_{xx} - 2k)u + \frac{1}{2} u_x^2\right) u_2 - \frac{1}{3} u_{ttt} - \frac{1}{3} u^3 u_{xxx} + \frac{1}{2} u_x (\frac{8}{3} u^3 + (-2u_{xxx} + k)u^2)$$

$$Q_5 = (-u + u_{xx}) u_t + u_{xtt} - u\text{^}3 u_{xxx} \frac{1}{2} u_x (9u^2 + 4\left(k - \frac{3}{2} u_{xx}\right) u)$$

$$Q_6 = u_t^2 - \frac{1}{2} u_x kt + \frac{u_{xxx}}{(-2u_{xx} + k + 2u)^{\frac{3}{2}}} + \frac{1}{2} k + u - \frac{u_x}{(-2u_{xx} + k + 2u)^{\frac{3}{2}}} \quad (16)$$

which $v_{Q_1}, v_{Q_2}, v_{Q_3}$ form a three-dimensional Lie algebra $\mathfrak{g}$ of symmetry group associated to the Camassa–Holm equation. The symmetries $u_x \partial_u$ and $u_t \partial_u$ of the Camassa-Holm equation are just the evolutionary representative of the space and time translational symmetry generators. Similarly the symmetry v_{Q_3} has geometric form $-kt\partial_x + 2t\partial_t + (-k - 2u)\partial_u$. We call these evolutionary symmetries v_{Q_1}, v_{Q_2} and v_{Q_3} in geometric form with

$$Y_1 = \partial_x, Y_2 = \partial_t, \quad Y_3 = -kt\partial_x + 2t\partial_t + (-k - 2u)\partial_u \quad (17)$$

respectively.

Proposition 2. The one-parameter groups $g_i(t): M \to M$ generated by the Y_i, $i = 1, 2, 3$ given in the following table:

$$g_1(s): \quad (x, t, u) \to (x + s, t, u)$$
$$g_2(s): \quad (x, t, u) \to (x, t + s, u)$$

$$g_3(s): \ (x, t, u) \to \left(-\tfrac{1}{2} kte^{2s} + \tfrac{1}{2} kt + x, te^{2s}, -\tfrac{1}{2} k + e^{-2s}\left(u + \tfrac{1}{2} k\right)\right) \quad (18)$$

where the entries give the transformed point $exp(sY_i)(x, t, u) = (\bar{x}, \bar{t}, \bar{u})$. Consequently, we can state the following theorem:

Theorem 3. If $u = U(x, t)$ is a solution of the equation (1) so are the functions $u^i(x.t), i = 1,2,3$ and $s \in R$, where

$$u^1 = U(x + s, t) \quad , \quad u^2 = U(x, t + s)$$
$$u^3 = e^{2s} U\left(\frac{kt}{2}(1 - e^{2s}) + x, te^{2s}\right) + \frac{k}{2}(1 - e^{-2s}) \quad (19)$$

OPTIMAL SYSTEM OF ONE-DIMENSIONAL SUBALGEBRAS

To each s-parameter subgroup there corresponds a family of group invariant solutions. Therefore, in general it is quite impossible to determine all possible group-invariant solutions of a PDE. In order to minimize this search, it is useful to construct the optimal system of solutions. It is well known that the problem of the construction of the optimal system of solutions is equivalent to that of the construction of the optimal system of subalgebras (Bluman, Cheviakov, & Anco, 2010; Olver, 2000). Here we will deal with the construction of the optimal system of subalgebras of $\mathfrak{g}$. Let G be a Lie group with $\mathfrak{g}$ its Lie algebra. Each element $T \in G$ yields Inner automorphism $T_a \to TT_a T^{-1}$ of the group G. Every automorphism of the group G induces an automorphism of $\mathfrak{g}$. The set of all these automorphisms is a Lie group called the adjoint group G^A. The Lie algebra of G^A is the adjoint algebra of $\mathfrak{g}$, defined as follows. Let us have two infinitesimal

generators $X, Y \in \mathfrak{g}$. The linear mapping $AdX: Y \to [X, Y]$ is an automorphism of $\mathfrak{g}$, called the inner derivation of $\mathfrak{g}$. The set of all inner derivations $adX\ (Y)\ (X, Y \in G)$ together with the Lie bracket $[AdX, AdY] = Ad[X, Y]$ is a Lie algebra $\mathfrak{g}^A$ called the adjoint algebra of $\mathfrak{g}$. Clearly $\mathfrak{g}^A$ is the Lie algebra of G^A.
Two subalgebras in $\mathfrak{g}$ are conjugate (or similar) if there is a transformation of G^A which takes one subalgebra into the other. The collection of pairwise non-conjugate s-dimensional subalgebras is the optimal system of subalgebras of order s. The latter problem tends to determine a list (that is, called an optimal system) of conjugacy inequivalent subalgebras with the property, that any other subalgebra is equivalent to a unique member of the list under some element of the adjoint representation i.e. $\bar{h}Ad(g)h$ for some g of a considered Lie group. Thus we will deal with the construction of the optimal system of subalgebras of $\mathfrak{g}$. The adjoint action has given by the Lie series

$$Ad\left(exp(\varepsilon Y_i)\, Y_j\right) = Y_j - \varepsilon\left[Y_i, Y_j\right] + \varepsilon^2\left[Y_i, \left[Y_i, Y_j\right]\right] - \cdots \tag{20}$$

where $\left[Y_i, Y_j\right]$ is the commutator for the Lie algebra, ε is a parameter and *i ,j = 1.2.3*. The adjoint representation of $\mathfrak{g}$ listed in the following table, it consists the separate adjoint actions of each element of $\mathfrak{g}$ on all other elements.

Table 1: Adjoint relations satisfied by infinitesimal generators

[,]	Y_1	Y_2	Y_3
Y_1	Y_1	Y_2	Y_3
Y_2	Y_1	Y_2	$Y_3 + \varepsilon(-kY_1{+}2Y_2)$
Y_3	Y_1	$Y_2 + \varepsilon(-kY_1{+}2Y_2)$	Y_3

where the (i,j)–th entry indicating $Ad\left(exp(\varepsilon Y_i)\, Y_j\right)$.

Theorem 4. An optimal system of one–dimensional Lie algebras of the Camassa-Holm equation (1) provided by

$$\begin{array}{ll} 1)\ Y_1 \quad 2) Y_2 & 3) Y_3 \\ 4)\ Y_3 + Y_2 & 5) Y_3 - Y_2 \\ 4)\ Y_3 + Y_1 & 5) Y_3 - Y_1 \end{array} \tag{21}$$

Proof. Consider the symmetry algebra $\mathfrak{g}$ of the Camassa–Holm equation whose adjoint representation was determined in table 1 and

$$Y = a_1Y_1 + a_2Y_2 + a_3Y_3 \tag{22}$$

is a nonzero vector filed in $\mathfrak{g}$. We will simplify as many of the coefficients a_i as possible through judicious applications of adjoint maps to Y. Suppose first that $a_3 \neq 0$. Scaling Y if necessary, we can assume that $a_3 = 1$. Referring to table 1, if we act on such a Y by $Ad\left(exp(-\frac{1}{k}a_1Y_2)\right)$, we can make the coefficient of Y_1 vanish.

$$\acute{Y} = Ad\left(exp\left(-\frac{1}{k}a_1Y_2\right)\right)Y = \acute{a}_2Y_2 + Y_3 \tag{23}$$

for certain scalar $\acute{a}_2$. So, depending on the sign of $\acute{a}_2$, we can make the coefficient of Y_2 either +1,-1 or 0. On the other hand, referring to table 1, if we act on Y by Ad (exp $(\frac{1}{2}a_2Y_2)$), we can cancel the coefficient of Y_2. So that Y is equivalent to

$$\acute{Y} = Ad\left(\exp\left(\frac{1}{2}a_2Y_2\right)\right)Y = \acute{a}_1Y_1 + Y_3 \tag{24}$$

For some $\acute{a}_1$. Thus, we can make the coefficient of Y_1 either +1,-1 or 0. Consequently, any one–dimensional subalgebra spanned by Y with $a_3 \neq 0$ is equivalent to one spanned by either $Y_3 + Y_2, Y_3 - Y_2, Y_3 + Y_{1,}Y_3 - Y_1Y_3$. The remaining cases, $a_3 = 0$, are similarly seen to be equivalent either to $Y_2(a_2 \neq 0)$ or to $Y_1(a_3 = a_2 = 0)$. There is not possible any further simplification. Recapitulating, we have found an optimal system of one–dimensional subalgebras to be those spanned by

$$a)Y_3 \quad b)Y_3 + Y_2 \quad C)Y_3 - Y_2 \quad d)Y_3 + Y_1$$
$$e)Y_3 - Y_1 \quad f)Y_2 \quad g)Y_1$$

HIGHER- ORDER CONSERVATION LAWS FOR THE CAMASSA-HOLM EQUATION

Consider a system of N partial differential equations of order n with p independent variables $x = (x^1, \ldots, x^p)$ and q dependent variables $u = (u^1, \ldots, u^q)$, given by PDE system (2). A Conservation law of a PDF system (2) is a divergence expression

$$D_1P_1 + \cdots D_pP_p = 0 \tag{25}$$

holding for all solutions $u = f(x)$ of the given system. In (26), $P_i\left(x, u^{(r)}\right), i = 1 \ldots p,$ are called the fluxes of the conservation law, and the highest–order derivative r present in the fluxes is called the order of the conservation law. If one of the independent variables of PDE system (2) is time t, the conservation law (25) takes the from

$$D_tT + DivX = 0 \tag{26}$$

where *Div* is the spatial divergence of *X* with respect to the spatial variables $x = (x^1, \ldots, x^p)$. Here T referred to as a density, and $X = (X_1, \ldots X_p)$ as spatial fluxes of the conservation law (25). The conserved density, T, and the associated flux, $X = (X_1, \ldots X_p)$ are functions of x, t, u and the derivatives of u with respect to both x and t. In particular, every admitted conservation law arises from multipliers $\lambda^v(x, u^{(1)})$ such that

$$\lambda^v\left(x, u^{(l)}\right)\Delta_v\left(x, u^{(n)}\right) = D_iP_i\left(x, u^{(r)}\right) \tag{27}$$

holds identically, where the summation convention is used whenever appropriate. Through this approach, the determining of conservation laws for a given PDE system (Bluman et al., 2010) (2) reduces to finding sets of multipliers. The Euler operator with respect to u_j is the operator defined by

$$E_{u^j} = \frac{\partial}{\partial_{u^j}} - D_i\frac{\partial}{\partial u_i^j} + \cdots + (-1)^sD_{i_1}\ldots D_{i_s}\frac{\partial}{\partial u_{i_1\ldots i_s}^j} + \cdots \tag{28}$$

It is well known that, the Euler operators (28) annihilate any divergence expression $D_iP_i\left(x, u^{(r)}\right)$. Thus, the following identities hold for arbitrary function u:

$$E_{u^j}\left(D_iP_i\left(x, u^{(r)}\right) = 0\,, \quad j = 1, \ldots q \tag{29}$$

The converse also holds. Specifically, the only scalar expressions annihilated by Euler operators are divergence expressions. In continuation, the following theorem is applied which connecting multipliers and conservation laws.

Theorem 5. A set of multipliers $\{\lambda^{v}(x, u^{(l)})\}_{v=1}^{N}$ yields a conservation law for the PDE system (2) if and only if the set of identities

$$E_{u^j}\left(\lambda^{v}(x,u^{(l)})\Delta_v(x,u^{(n)})\right)=0,\quad j=1,\ldots,q \tag{30}$$

holds identically. See [3] for more details. The set of equations (30) yields the set of linear determining equations to find all sets of conservation law multipliers of the PDE system (2) by considering multipliers of all orders.

In this section, we construct higher order conservation laws for the Camassa-Holm equation. Consider the multipliers of the form $\lambda(x,t,u,u_x,u_t,u_{xx},u_{tt},u_{xt},u_{xxx},u_{ttt})$ for the equation (1). The determining equation for multipliers is

$$E_u\left[\lambda(u_t-u_{tx^2}+ku_x+3_{uu_x}-2u_xu_{x^2}-uu_{x^3}\right]=0 \tag{31}$$

In (31) E_u is the standard Euler operator defined by

$$E_u=\frac{\partial}{\partial u}-D_x\frac{\partial}{\partial_{u_x}}-D_t\frac{\partial}{\partial_{u_t}}+D_x^2\frac{\partial}{\partial_{u_{xx}}}+D_xD_t\frac{\partial}{\partial_{u_{xt}}}+D_t^2\frac{\partial}{\partial_{u_{tt}}}-\cdots, \tag{32}$$

where D_x and D_t are the total derivatives with respect to x and t. Therefore, after straightforward but tedious calculation, we conclude that

$$\lambda=C_1u_{tt}+C_1u_xu_t+(-C_1u+C_2)\left(u_{xt}+\frac{1}{2}{u_x}^2\right)+\frac{2C_5}{\sqrt{-u_{xx}+u}}+(-2C_1u+C_2)uu_{xx}+\frac{5}{2}C_1u^3-\frac{3}{2}C_2u^2+C_3u+C_4 \tag{33}$$

where C_1, C_2, C_3, C_4, C_5 are constants. To calculate the conserved quantities T and X, we need to invert the total divergence operator. This requires the integration (by parts) of an expression in multi–dimensions involving arbitrary functions and its derivatives, which is a difficult and cumbersome task. The homotopy operator (10) is a powerful algorithmic tool (explicit formula) that originates from homological algebra and variational bi–complexes. In the following, conserved vectors represented by two components T_1 and T_2 , which are conserved density and flux, respectively. By using the 2-dimensionl homotopy (integral) formula which is due to Hereman et al. (2005), we derive the following conserved vectors:

Theorem 6. Conservation laws of the Camassa-Holm equation obtained as follows

Case 1 : $\lambda=u_{tt}+u_tu_x-uu_{tx}-\frac{1}{2}uu_x^2-2u_{xx}u^2+\frac{5}{2}u^3$

Therefore, we obtain the following conserved vector

$$\begin{aligned}T_1=&\frac{1}{24}u^3u_{xxxx}-\frac{7}{6}u_{tt}u^2+\frac{3}{8}u^2u_x^2+\frac{5}{18}u^2u_{txxx}\\&+\frac{1}{3}uu_{ttxx}+\frac{1}{6}u_xu_{ttx}-\frac{1}{6}u_{tt}u_{xx}\\&-\frac{1}{2}u_tu_{txx}+\frac{1}{3}u^2u_{xx}^2-\frac{23}{24}u^3u_{xx}-\frac{1}{24}u_x^4\\&+\frac{5}{8}u^4+\frac{1}{2}u_t^2-\frac{1}{3}u_{xx}u_x^2u+\frac{2}{3}u_xu_tu\\&-\frac{1}{9}u_tuu_{xxx}+\frac{1}{3}uu_{txx}u_x+\frac{2}{3}u_{tx}u_{xx}u\\&-\frac{1}{24}u^2u_{xxx}u_x-\frac{2}{3}u_xu_tu_{xx}\end{aligned} \tag{34}$$

$$\begin{aligned}T_2=&\frac{1}{6}u^2u_xu_{txx}-\frac{4}{9}uu_tu_{txx}+\frac{1}{2}uu_x^2u_{tx}-\frac{1}{8}u^2u_{xxx}u_t\\&+\frac{7}{8}u^2u_xu_t-\frac{8}{9}uu_{tt}u_{xx}+\frac{1}{3}uu_t^2\\&+\frac{7}{6}u^2u_{tt}-\frac{1}{3}uu_{ttx}-\frac{1}{24}u^3u_{txxx}\\&+\frac{1}{6}u_xu_{ttt}+\frac{1}{6}u_tu_{ttx}-\frac{1}{3}u_{tx}u_{tt}-\frac{1}{3}u_x^3u_t\\&-\frac{2}{9}u_x^2u_{tt}+\frac{1}{9}u_t^2u_{xx}-\frac{5}{18}u^2u_{ttxx}\\&+\frac{2}{9}uu_{tx}^2-\frac{1}{2}u_x^2u^3-\frac{37}{24}u_{tx}u^3-\frac{5}{2}u^4u_{xx}\\&+u^3u_{xx}^2+\frac{2}{9}uu_xu_{xxxx}-\frac{2}{9}u_xu_tu_{tx}\\&+\frac{7}{6}u_{tx}u_{xx}u^2+\frac{1}{2}u_x^2u_{xx}u^2-\frac{2}{3}uu_xu_tu_{xx}\\&+\frac{3}{2}u^5\end{aligned} \tag{35}$$

Case 2 : $\lambda=u_{tx}+\frac{1}{2}u_x^2+uu_{xx}-\frac{3}{2}u^2$

Therefore, the corresponding conserved vector is:

$$\begin{aligned}T_1=&-\frac{1}{2}u^3-\frac{1}{6}u_{xx}u_x^2+\frac{1}{4}u_tu_x+\frac{1}{3}u^2u_{xx}+\frac{1}{6}uu_x^2\\&-\frac{1}{12}u_xu_{txx}+\frac{1}{18}u^2u_{xxxx}+\frac{1}{12}uu_{txxx}\\&+\frac{1}{4}uu_{tx}-\frac{1}{6}u_{tx}u_{xx}+\frac{1}{9}uu_xu_{xxx}\end{aligned} \tag{36}$$

$$\begin{aligned}T_2=&-\frac{1}{9}uu_tu_{xxx}-\frac{1}{2}uu_{ttxx}-\frac{1}{18}u^2u_{txxx}+\frac{1}{12}uu_{xx}u_x^2\\&+\frac{1}{4}u_t^2-\frac{1}{3}u_{tx}^2-\frac{9}{8}u^4-\frac{1}{8}u_x^4+\frac{1}{3}uu_tu_x\\&+uu_{tx}u_{xx}-\frac{1}{4}uu_{tt}+\frac{3}{4}u^2u_x^2-\frac{1}{12}u_tu_{txx}\\&+\frac{7}{6}u^2u_{tx}+\frac{1}{6}u_xu_{ttx}+\frac{3}{2}u^3u_{xx}-\frac{1}{3}u_x^2u_{tx}\\&-\frac{1}{2}u^2u_{xx}^2-\frac{1}{2}u_{xx}u_x^2u\end{aligned} \tag{37}$$

Case 3 : $\lambda=\frac{2}{\sqrt{-u_{xx}+u}}$

$$\begin{aligned}T_1=&\frac{1}{3}(2u_{xx}u_x^2+18uu_{xx}^2-4u_{xx}^3-26u_{xx}u^2-3uu_{xxx}^2\\&+2u_{xx}uu_{xxxx}-5uu_x^2-2u^2u_{xxxx}\\&+8u_xuu_{xxx}+12u^3\\&-2u_{xxx}u_{xx}u_x)/(-u_{xx}+u)^{\frac{5}{2}}\end{aligned} \tag{38}$$

$$T_2 = \frac{1}{3}(12u^4 - 12uu_{xx}^3 - 7uu_{txx}u_x + 3uu_{txx}u_{xxx} + 5u_xu_tu - u_tuu_{xxx} + 18u_{tx}u_{xx}u - 2u_xu_tu_{xx} + 4u_xu_{txx}{u_x}^2 - 2u_tu_{xxx}u_{xx} - 2uu_{txxx}u_{xx} + 36u^2u_{xx}^2 - 36u^3u_{xx} + 2u^2u_{txxx} - 8u_{tx}u_{xx}^2 - 10u_{tx}u^2)/(-u_{xx} + u)^{\frac{5}{2}} \quad (39)$$

Case 4 : $\lambda = u$

$$T_1 = \tfrac{1}{6}u_x^2 - \tfrac{1}{3}uu_{xx} + \tfrac{1}{2}u^2 \quad (40)$$

$$T_2 = \tfrac{1}{3}u_tu_x - \tfrac{2}{3}uu_{tx} + u^3 - u^2u_{xx}(41) \quad (41)$$

Case 5: $\lambda = 1$

$$T_1 = u - \tfrac{1}{3}u_{xx} \quad (42)$$

$$T_2 = -uu_{xx} - \tfrac{2}{3}u_{tx} + \tfrac{3}{2}u^2 - \tfrac{1}{2}{u_x}^2 \quad (43)$$

CONCLUSIONS

In this paper, generalized symmetries of order three of the Camassa-Holm equation obtained by the infinitesimal criterion method. Classification of one-dimensional subalgebras is determined by constructing one-dimensional optimal system of Lie subalgebras. Furthermore, we utilize Euler operator to construct determining equation for multipliers. Finally, higher order conservation laws of the Camassa-Holm equation constructed by applying the 2-dimensional Homotopy formula.

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