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Multiple positive solutions for some P-Laplacian nonlinear problem at infinity

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ABSTRACT

In recent years, boundary value problem of second-order have received a lot of attention. In this paper, I study the existence of positive solutions for a class of p-Laplacian boundary value problem at infinity. The fixed point theorems in cones is the our main tools to prove the existence of solutions. I provide sufficient conditions under which this system has solution. I establish some propositions to prove the existence of positive solutions for these equations.

Keywords: Boundary value problem, Positive solution, (p,q) -Laplacian system, Fixed point

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INTRODUCTION

In recent years, boundary value problems have received a lot of attention. For example (Liang & Zhang, 2009; Pang, Lian, & Ge, 2007) have studied the existence of positive solutions for some boundary value problems.
In this paper, we study the existence of positive solutions for the following system:

$$\begin{cases}(\phi_p(u'))' + m(t)f(u,v) = 0 \quad , \\ (\phi_p\ (v'))' + n(t)g(u,v) = 0\end{cases} \tag{1}$$

$$\begin{cases}u(0) - \alpha_0(u'(\eta)) = 0, \quad u'(+\infty) = 0, \\ v(0) - \beta_0(v'(\xi)) = 0, \quad v'(+\infty) = 0\end{cases} \tag{2}$$

Where

$$\phi_p(s) = |s|^{p-2}s \ , \ p > 1, \phi_q = (\phi_p)^{-1}, \frac{1}{p} + \frac{1}{q} = 1,$$

$\eta, \xi \in (0,+\infty), m, n : [0,+\infty) \to [0,+\infty)$ have countably many singularities on $[0,+\infty)$. α_0, β_0 are functions which satisfy the conditions that there are nonnegative $\alpha_1, \beta_1, \alpha_2,\ \beta_2$ such that

$$\alpha_1 x \le \alpha_0(x) \le \alpha_2 x \ , \ \beta_1 x \le \beta_0(x) \le \beta_2 x \quad \text{for} \quad x, y \in \Re .$$

Liang & Zhang, (2009), studied the existence of positive solutions for

$$\begin{cases}(\phi\ (u'(t)))' + a(t)f(t,u(t)) = 0, 0 \in [0,+\infty) \\ u(0) - B_0 u'(\eta) = 0, \quad u'(+\infty) = 0\end{cases}$$

Where $\phi(s) : \Re \to \Re$ is an increasing homomorphism and $\phi(0) = 0$. $\eta \in (0,+\infty), a : [0,+\infty) \to [0,+\infty)$.

Now, we assume that the following conditions:

H_1) $f, g \in C([0,+\infty)^2, [0,+\infty))$, $f(0,0) \ne 0, g(0,0) \ne 0$ on any subinterval of $[0,+\infty)$ and when u, v are bounded, $f((1+t)u,(1+t)v)$, $g((1+t)u,(1+t)v)$ are bounded on $[0,+\infty)^2$.

H_2) There exists a sequence $\{t_i\}_{i=1}^{\infty}$ such that $1 \le t_{i+1} \le t_i$, $\lim_{i\to+\infty} t_i = t_0 < \infty$, $t_0 > 1$, $\lim_{t\to t_i} m(t) = \infty$, $i = 1,2,\cdots$, and

$$\int_0^{+\infty} \phi_p^{-1}\left(\int_s^{+\infty} m(t)dt\right)ds < +\infty$$

$$\int_0^{+\infty} \phi_p^{-1}\left(\int_s^{+\infty} n(t)dt\right)ds < +\infty , \tag{3}$$

H_3) There exists a sequence $\{t_i\}_{i=1}^{\infty}$ such that $0 < t_{i+1} < t_i < 1$, $\lim_{i\to+\infty} t_i = t_0 < \infty$, $t_0 > 1$, $\lim_{t\to t_i} a(t) = \infty$, $i = 1,2,\cdots$ and (3) holds.

SOME DEFINITIONS AND FIXED POIN THEOREMS

Definition (1)
Let $(X, \|.\|)$ be a real Banach space and a non-empty, closed, convex C subset of X is called a Cone of X, If it satisfies the following conditions:
i) If $x \in C$ and $\lambda \geq 0$ implies that $\lambda x \in C$, ii) If $x \in C$ and $-x \in C$ implies that $x = 0$,
Every cone C subset of X includes an ordering in X which is given by $x \leq y$ if and only if $y - x \in C$.

Definition (2)

A map $\psi: P \to [0, +\infty)$ is called nonnegative continuous concave functional provided ψ is nonnegative, continuous and satisfies,

$$\psi(tx + (1-t)y) \geq t\psi(x) + (1-t)\,\psi(y)$$

for all $x, y \in P$ and $t \in [0,1]$.
Similarly, we say the map β is a nonnegative continuous convex functional on a cone P of X $\beta: P \to [0, +\infty)$ is continuous and $\beta(tx + (1-t)y) \geq t\beta(x) + (1-t)\,\beta(y)$ for all $x, y \in P$ and $t \in [0,1]$. The main tool of this paper is the following fixed-point:

Theorem (3) (Deimling, 2010) **.**
Let E be a Banach space and P subset of E be a cone in E. let $r > 0$ define $\Omega_r = \{x \in P \mid \|x\| < r\}$. Assume that $T: P \cap \overline{\Omega_r} \to P$ is completely continuous operator such that $Tx \neq x$ for $x \in \partial\Omega_r$,

i) If $\|Tu\| \leq \|u\|$ for $u \in \partial\Omega_r$ then $i(T, \Omega_r, P) = 1$

ii) If $\|Tu\| \geq \|u\|$ for $u \in \partial\Omega_r$ then $i(T, \Omega_r, P) = 0$

PRELIMINARIES AND LEMMAS
Let,

$$E = \left\{(u,v) \in c[0,+\infty) \times c[0,+\infty) \,\middle|\, sup_{\,0\leq t} \frac{|u(t)|}{1+t} < \infty \,,\; sup_{0\leq t} \frac{|u(t)|}{1+t} < \infty \right.$$

Then E is a banach space with the norm $\|(u,v)\| = \|u\| + \|v\|$ where $\|u\| = sup_{0\leq t} \frac{|u(t)|}{1+t} < +\infty$.

Define cone K subset of E by

$$K = \left\{(u,v) \in E \,\middle|\, u, v \text{ are concaves}, \lim_{t\to\infty} u'(t) = 0, \lim_{t\to\infty} v'(t) = 0, \right\}$$

Lemma (4) (Liang & Zhang, 2009)**.**
Suppose H_2 holds. Then for any $\gamma \in (1, +\infty)$ which satisfies $0 < \int_{\frac{1}{\gamma}}^{\gamma} m(t)dt < +\infty$, $0 < \int_{\frac{1}{\gamma}}^{\gamma} n(t)dt < +\infty$, and the functions

$$K_1(t) = \int_{\frac{1}{s_0}}^{t} \phi_p^{-1}\left(\int_s^{s_0} m(\rho)d\rho\right)ds + \alpha_1 \phi_p^{-1}\left(\int_t^{s_0} n(\rho)d\rho\right)$$

$$K_2(t) = \int_{\frac{1}{s_0}}^{t} \phi_p^{-1}\left(\int_s^{s_0} n(\rho)d\rho\right)ds + \beta_1 \phi_p^{-1}\left(\int_t^{s_0} n(\rho)d\rho\right)$$

Are continuous and positive on $[\frac{1}{s_0}, s_0]$.
In addition $H_i = \min_{t\in[\frac{1}{s_0}, s_0]} K_i(t) > 0,\; i = 1,2.$

Lemma (5) (Liang & Zhang, 2009).
Let u be a nonnegative concave function on $[0, +\infty)$ and $\lim_{t\to\infty} u'(t) = 0$, and [a , b] be a subset of $(0, +\infty)$.

Then $u(t) \geq \lambda(t)\|u\|$ where $\lambda(t) = \{^{\sigma, t\geq\sigma}_{t, t\leq\sigma}$ and
$\sigma = \inf\{ \xi \in [0, +\infty): sup_{0\leq t<+\infty} \frac{|u(t)|}{1+t} = \frac{|u(\xi)|}{1+\xi} \}$
Now, we define an operator

$$T: K \to c[0,+\infty) \times c[0,+\infty)$$
$$T(u,v)(t) = (T_1(u,v), T_2(u,v))(t)$$

Such that

$$T_1(u,v) = \int_0^t \phi_p^{-1}\left(\int_s^{+\infty} m(\rho) f(u(\rho), v(\rho))d\rho\right)ds + \alpha_0 \phi_p^{-1}\left(\int_t^{s_0} m(\rho) f(u(\rho), v(\rho))d\rho\right), \qquad (4)$$

$$T_2(u,v) = \int_0^t \phi_p^{-1}\left(\int_s^{+\infty} n(\rho) g(u(\rho), v(\rho))d\rho\right)ds + \beta_0 \phi_p^{-1}\left(\int_t^{s_0} n(\rho) g(u(\rho), v(\rho))d\rho\right), \qquad (5)$$

Lemma (6) (Liu, 2003)
Let W be a bounded subset of K. Then W is relatively compact in E if $\left\{\frac{w(t)}{1+t}\right\}$ are equicontinuous on any finite subinterval of $[0, +\infty)$ and for any $\varepsilon > 0$ there exists $N > 0$ such that $\left|\frac{x(t_1)}{1+t_1} - \frac{x(t_2)}{1+t_2}\right| < \varepsilon$, uniformly with respect to $x \in W$ as $t_1, t_2 \geq N$, where $W(t) = \{x(t): x \in W\}$, $t \in [0, +\infty)$.

Lemma (7)
Let H_1, H_2, H_3 hold. Then $T: K \to K$ is completely continuous.

MAIN RESULT

Theorem (8)
Suppose that H_1, H_2, H_3 hold. Let $\{\lambda_k\}_{k=1}^{+\infty}$ such that $\lambda_k \in (t_k, t_{k+1})$, $k = 1,2,\cdots$. Let $\{m_k\}_{k=1}^{+\infty}$ and $\{M_k\}_{k=1}^{+\infty}$ be such that $M_{k+1} < \frac{\Gamma(\frac{1}{\lambda_k})}{1+\lambda_k} m_k < m_k < q m_k < M_k$, and for $k \in N$, we assume that f, g satisfy,

***H*$_4$**) $f((1+t)u,(1+t)v) \geq \phi_p(qm_k)$, $g((1+t)u,(1+t)v) \geq \phi_p(qm_k)$

For

$$(t,u,v) \in \left[\frac{1}{\lambda_k},\ \lambda_k\right] \times \left[\frac{\Gamma\left(\frac{1}{\lambda_k}\right)}{1+\lambda_k} m_k, m_k\right] \times \left[\frac{\Gamma\left(\frac{1}{\lambda_k}\right)}{1+\lambda_k} m_k, m_k\right].$$

***H*$_5$**) $f((1+t)u,(1+t)v) \geq \phi_p(QM_k)$, $g((1+t)u,(1+t)v) \geq \phi_p(QM_k)$,

For $(t,u,v) \in [0,+\infty] \times [0, M_k] \times [0, M_k]$

where

$$q \in (\Gamma_1, +\infty), Q \in (0, \Gamma_2)\ , \quad \Gamma_1 = \frac{1+t_0}{L}\ , L > 0,$$

$$\Gamma_2 = \frac{1}{\max(\phi_p^{-1}(\int_0^{+\infty} m(\rho)d\rho)(1+\alpha_2),\ \phi_p^{-1}(\int_0^{+\infty} n(\rho)d\rho)(1+\beta_2))}$$

Then the boundary value system (1) and (2) has infinitely many solutions $\{(u_k,v_k)\}_{k=1}^{+\infty}$ such that $m_k \leq \|(u_k,v_k)\| \leq M_k$, $k = 1,2,\cdots$.

Proof. We assume that the sequence $\{\Omega_{1k}\}_{k=1}^{+\infty}$ and $\{\Omega_{2k}\}_{k=1}^{+\infty}$ of open subsets of E be as following:

$\Omega_{1k} = \{(u,v) \in K \mid \|(u,v)\| < 2m_k\}$,

$\Omega_{2k} = \{(u,v) \in K \mid \|(u,v)\| < 2M_k\}, \quad k = 1,2,\cdots$.

We know that $1 < t_0 \leq t_{k+1} < \lambda_k < t_k < +\infty, k = 1,2,\cdots$, so from lemma (5) for $k \in N$ and $u,v \in K$ we have $u(t) \geq \Gamma(t)\|u\|$, $t \in \left[\frac{1}{\lambda_k},\ \lambda_k\right]$.

Let $k \in N$ and $(u,v) \in \partial\Omega_{1k}$, then we have

$$2m_k = \|(u,v)\| = sup_{t\geq 0}\frac{|u(t)|}{1+t} + sup_{t\geq 0}\frac{|v(t)|}{1+t} \geq \frac{\left|u(\frac{1}{\lambda_k})\right|}{1+\lambda_k} + \frac{\left|v(\frac{1}{\lambda_k})\right|}{1+\lambda_k}$$
$$\geq \frac{\Gamma\left(\frac{1}{\lambda_k}\right)}{1+\lambda_k}(\|(u,v)\|),\ t \in \left[\frac{1}{\lambda_k},\ \lambda_k\right].$$

From (H_4) we have $f((1+t)u,(1+t)v) \geq \phi_{p_1}(qm_k)$, we know that $\left(\frac{1}{t_0}, t_0\right) \subseteqq \left[\frac{1}{\lambda_k},\ \lambda_k\right]$, if (H_2) holds, we consider three cases:

i) If $\eta \in \left[\frac{1}{t_0}, t_0\right]$: we have;

$$\|T_1(u,v)\|$$
$$= sup_{t\geq 0}\frac{1}{1+t}\left|\begin{matrix}\int_0^t \phi_p^{-1}(\int_s^{+\infty} m(\rho)f(u(\rho),v(\rho))d\rho)ds \\ +\alpha_0\phi_p^{-1}(\int_\eta^{+\infty} m(\rho)f(u(\rho),v(\rho))d\rho)\end{matrix}\right|$$
$$\geq \frac{1}{1+t_0}(qm_k)\int_{\frac{1}{t_0}}^{\eta}\phi_p^{-1}(\int_s^{t_0} m(\rho)\,d\rho)ds$$
$$+\alpha_1\phi_p^{-1}(\int_\eta^{t_0} m(\rho)d\rho)$$
$$= \frac{qm_k}{1+t_0}K_1(\eta) > \frac{Lqm_k}{1+t_0} > 2m_k = \|(u,v)\|.$$

ii) If $\eta \in (0, \frac{1}{t_0})$ from (4) and (H_4) and lemma (4) we see :

$\|T_1(u,v)\| =$

$sup_{t\geq 0}\frac{1}{1+t}\left|\int_0^t \phi_p^{-1}(\int_s^{+\infty} m(\rho)f(u(\rho),v(\rho))d\rho)ds + \alpha_0\phi_p^{-1}(\int_\eta^{+\infty} m(\rho)f(u(\rho),v(\rho))d\rho)\right|$

$$sup_{t\geq 0}\frac{1}{1+t}\alpha_1\phi_p^{-1}(\int_{\frac{1}{t_0}}^{t_0} m(\rho)f(u(\rho),v(\rho))d\rho)$$
$$\geq \frac{qm_k}{1+t_0}\alpha_1\phi_p^{-1}(\int_{\frac{1}{t_0}}^{t_0} m(\rho)\,d\rho)$$

$= \frac{qm_k}{1+t_0}K_1\left(\frac{1}{t_0}\right) > \frac{Lqm_k}{1+t_0} > 2m_k = \|(u,v)\|.$

iii) If $\eta \in (t_0, +\infty)$. From (4) , (H_4) and lemma (4) we have

$$\|T_1(u,v)\|$$
$$= sup_{t\geq 0}\frac{1}{1+t}\left|\int_0^t \phi_p^{-1}(\int_s^{+\infty} m(\rho)f(u(\rho),v(\rho))d\rho)ds\right.$$
$$\left.+\alpha_0\phi_p^{-1}(\int_\eta^{+\infty} m(\rho)f(u(\rho),v(\rho))d\rho)\right| \geq \frac{qm_k}{1+t_0}K_1(t_0)$$
$$> \frac{Lqm_k}{1+t_0} > 2m_k = \|(u,v)\|.$$

Since

$$\|T(u,v)\| = \|T_1(u,v)\| + \|T_2(u,v)\| \geq \|(u,v)\|\ ,$$

so from theorem (3) implies that

$$\mathrm{i}(\mathrm{T}, \Omega_{1k}, K) = 0, \qquad (6)$$

Suppose that $(u,v) \in \partial\Omega_{2k}$ and $u,v \in [0, \mathrm{M}_k]$. Thus

$$\frac{u(t)}{1+t} \leq sup_{t\geq 0}\frac{|u(t)|}{1+t} \leq \|u\| = M_k\ ,$$
$$\frac{v(t)}{1+t} \leq sup_{t\geq 0}\frac{|v(t)|}{1+t} \leq \|v\| = M_k$$

From (H_4) we have

$$f((1+t)u,(1+t)v) \leq \phi_p(QM_k),$$

so

$\|T_1(u,v)\| =$

$sup_{t\geq 0}\frac{1}{1+t}\left|\int_0^t \phi_p^{-1}(\int_s^{+\infty} m(\rho)f(u(\rho),v(\rho))d\rho)ds + \alpha_0\phi_p^{-1}(\int_\eta^{+\infty} m(\rho)f(u(\rho),v(\rho))d\rho)\right| \leq (1+\alpha_2)\phi_p^{-1}(\int_0^{+\infty} m(\rho)f(u(\rho),v(\rho))d\rho) \leq QM_k(1+\alpha_2)\phi_p^{-1}(\int_0^{+\infty} m(\rho)\,d\rho) \leq M_k = \|u\| = \frac{1}{2}\|(u,v)\|$.

Similarly we can see $\|T_2(u,v)\| \leq \|v\| = \frac{1}{2}\|(u,v)\|$.

Then

$$\|T(u,v)\| = \|T_1(u,v)\| + \|T_2(u,v)\| \leq \|(u,v)\|$$

for $(u,v) \in \partial\Omega_{2k}$, Hence , from theorem (3) we have

$$\mathrm{i}(\mathrm{T}, \Omega_{2k}, K) = 1, \qquad (7)$$

Thus from additivity the fixed-point index we have

$$\mathrm{i}(\mathrm{T}, \Omega_{2k} \backslash \overline{\Omega_{1k}}, K) = 1$$

and T has a fixed point in $\Omega_{2k} \backslash \overline{\Omega_{1k}}$ such that

$$m_k \leq \|(u,v)\| \leq M_k \text{ , for } k \in N$$

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