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R.S. Torogh
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Existence of Positive Solutions for (p_1, p_2)- Laplacian System to Dirichlet Boundary Conditions

R. Sahandi Torogh

Department of Mathematics, Varamin-Pishva branch, Islamic Azad University, Varamin, Iran

sahandi@iauvaramin.ac.ir

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ABSTRACT

In this paper, we deal with the multiplicity of positive solutions for a class of (p , q)-Lalpacian system. Moreover the author established suitable conditions under which, the problem has positive solutions.

Keywords: Positive solution; lower and upper solution; (p_1, p_2)-Laplacian system

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INTRODUCTION

In this paper, we study the existence of positive solutions for the following (p_1, p_2)-Laplacian system.

$$\begin{cases} \varphi_{p_1}(u_1{}') + h_1(t) f_1(u_1, u_2) = 0, t \in (0,1) \\ \varphi_{p_2}(u_2{}') + h_2(t) f_2(u_1, u_2) = 0, \\ u_1(0) = u_1(1) = u_2(0) = u_2(0). \end{cases} \tag{1}$$

Where $\varphi_{p_i}(x) = |x|^{p_i - 2} x$, $p_i > 1, x \in \varphi_{p_1}(u_1{}')$, $f_i \in C([0,1] \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$, $\lambda, \mu > 0$, $f_i(0,0,0) = 0$, for i=1,2, $h_i \in L^1_{loc}$, $h_i \in X_i$, where

$$X_i = \left\{ h_i \in L^1_{loc} \middle| \begin{array}{l} \int_0^{\frac{1}{2}} \varphi_{p_i}^{-1} (\int_s^{\frac{1}{2}} |h_i(r)| dr) ds + \\ \int_{\frac{1}{2}}^{s} \varphi_{p_i}^{-1} (\int_{\frac{1}{2}}^{s} |h_i(r)| dr) ds \ < \infty \end{array} \right\} \tag{2}$$

In recent years, many authors have studied the existence of positive solutions for boundary value problems. For example (Liang & Zhang, 2009; Pang, Lian, & Ge, 2007; Sun & Ge, 2007) have studied the existence of positive solutions for some boundary value problems. The existence of three positive solutions for the problem,

$$\begin{array}{c} (\varphi_p(u'(x)))' + \lambda h(x) f(u(x)) = 0, x \in (0,1) \\ u(0) = u(1) = 0. \ * \end{array}$$

was studied by (Sim & Tanaka, 2015). Cheng and Lü (2012) studied the existence of solutions for some nonlinear eigenvalue (p,q)-Laplacian system. (Lee, Kim, & Lee, 2014) have studied the existence of solutions for ,

$$\begin{array}{c} (\varphi_p(u'(x)))' + h(x) f(u(x)) = 0 \quad , x \in (0,1) \\ u(0) = u(1) = 0. \ ** \end{array}$$

In this paper we extend the existence result of $(*), (**)$ to the problem (1).

PRELIMINARIES

Let

$$C_2[0,1] = \left\{(u_1,u_2) \in (C[0,1] \times C[0,1]) \cap (C^1[0,1] \times C^1[0,1] \middle| -\infty < \lim_{t\to 0^+} m_i(t)u_i'(t) < \infty,\right\}$$

Where

$$m_i h_i(t) = \begin{cases} \varphi_{p_i}^{-1}\left(\int_{\frac{1}{2}}^{t} |h_i(s)|ds)\right)^{-1}, 0 \le t \le \frac{1}{2} \\ \varphi_{p_i}^{-1}\left(\int_{\frac{1}{2}}^{t} |h_i(s)|ds)\right)^{-1}, \frac{1}{2} \le t \le 1 \end{cases} \quad (3)$$

$m_i(t) = min\{m_i h_i(t), 1\}$, (4).

We define

$$\|u_i\|_{m_i} = \|u_i\|_\infty + \|m_i u_i'\|_\infty$$

Let $\|(u_1,u_2)\|_2 = \|u_1\|_{m_1} + \|u_2\|_{m_2}$. So $(C_2[0,1], \| \quad \|_2)$ is a Banach space.
Let $K = \{(u_1,u_2) \in C_2[0,1] | u_i(0) = u_i(1) = 0, i = 1,2\}$ and

$$u_i(t) = \begin{cases} \int_0^t \varphi_{p_i}^{-1}\left(\varphi_{p_i}\left(u_i'\left(\frac{1}{2}\right)\right) - \int_s^{\frac{1}{2}} \varphi_{p_i}\left(u_i'(r)\right)' dr\right) ds, \\ \qquad 0 \le t \le \frac{1}{2} \\ \int_t^1 \varphi_{p_i}^{-1}\left(-\varphi_{p_i}\left(u_i'\left(\frac{1}{2}\right)\right) - \int_{\frac{1}{2}}^{s} \varphi_{p_i}\left(u_i'(r)\right)' dr\right) ds, \\ \qquad \frac{1}{2} \le t \le 1 \end{cases} \quad (4)$$

Then $(u_1(t), u_2(t))$ is a solution of problem (1). We define

$$T_i(u_1(t),u_2(t)) = \begin{cases} \int_0^t \varphi_{p_i}^{-1}\left(\varphi_{p_i}\left(u_i'\left(\frac{1}{2}\right)\right) h_i f_i(u_1,u_2)) + \int_s^{\frac{1}{2}} h_i(r) f_i(u_1(r),u_2(r))\, dr\right) ds, \\ \qquad 0 \le t \le \frac{1}{2}\ , \\ \int_t^1 \varphi_{p_i}^{-1}\left(-\varphi_{p_i}\left(u_i'\left(\frac{1}{2}\right)\right) h_i f_i(u_1,u_2)) + \int_{\frac{1}{2}}^{s} h_i(r) f_i(u_1(r),u_2(r)\, dr\right) ds, \\ \qquad \frac{1}{2} \le t \le 1 \end{cases} \quad (5)$$

And $T(u_1,u_2) = (T_1(u_1,u_2), T_2(u_1,u_2)), (u_1,u_2) \in K$. Then $T(u_1,u_2) = (u_1,u_2)$ if and only if (u_1,u_2) is a solution of problem (1).

Theorem (1) Operator $T: K \to K$ is completely continuous.

Proof. Suppose $G_1 \times G_2$ be a bounded subset of K. We prove $T(u_{1n},u_{2n})$ is relative compact for a sequence $\{(u_{1n},u_{2n})\} \subset G_1 \times G_2$. We show that if $(u_{01},u_{02}) \in K$ and there is a subsequence $T(u_{1nl},u_{2nl})$ of $T(u_{1n},u_{2n})$ such that $T(u_{1nl},u_{2nl}) \to (u_{01},u_{02})$ where $l \to \infty$ in K and T is continuous on K. We prove $\{(m_1 T_1(u_{1n},u_{2n}), (m_2 T_2(u_{1n},u_{2n})\}$ is uniform bounded in $C_2[0,1]$.
Since $G_1 \times G_2$ is bounded in K, there exists $\lambda_G > 0$. Such that $\|u_1\|_\infty + \|u_2\|_\infty < \lambda_G$ and $\|m_1 u_1'\|_\infty + \|m_2 u_2'\|_\infty < \lambda_G$, for $(u_1,u_1) \in G_1 \times G_2$. There is $\lambda_G' > 0$ so that $\left|\varphi_{p_i}\left(u_i'\left(\frac{1}{2}\right)\right)(h_i f_i(u_1,u_2))\right| < \lambda_G'$.
Similar to (Lee et al., 2014), we have

$$|(T_i(u_{1n},u_{2n}))'(t)| \le$$
$$\varphi_{p_i}^{-1}\left(\left|\varphi_{p_i}\left(\varphi_{p_i} u_i'\left(\frac{1}{2}\right)\right)(h_i f_i(u_1,u_2))\right| + \int_t^{\frac{1}{2}} |h_i(s)||f_i(u_{1n}(s),u_{2n}(s))|\, ds\right)$$
$$\le \varphi_{p_i}^{-1}(\lambda_G' + \overline{f_i} \int_t^{\frac{1}{2}} |h_i(s)|ds \le \lambda_{p_i}(\varphi_{p_i}^{-1}(\lambda_G')$$
$$+ \varphi_{p_i}^{-1}(\overline{f_i})\varphi_{p_i}^{-1}\left(\int_t^{\frac{1}{2}} |h_i(s)|ds)\right) \quad (6)$$

Where

$$\overline{f_i} = \max_{s\in[-\lambda_G,\lambda_G]} |f_i(u_1(s),u_2(s))|\, \lambda_{p_i} = \begin{cases} 1\ , & p_i > 2 \\ 2^{(2-p_i)(p_i-1)}, & 1 < p_i \le 2 \end{cases} \quad (7)$$

So,

$$m_i(t)\varphi_{p_i}^{-1}\left(\int_t^{\frac{1}{2}} |h_i(s)|ds\right) \le 1, \quad (8)$$

then for $t \in \left(0, \frac{1}{2}\right)$,

$$m_i(t)\left|(T_i(u_{1n}(t),u_{2n}(t)))'\right| \le \lambda_{p_i}(\varphi_{p_i}^{-1}(\lambda_G') + \varphi_{p_i}^{-1}(\overline{f_i})) \quad (9)$$

thus,

$$|(m_1(T_1(u_{1n},u_{2n}))',m_2(T_2(u_{1n},u_{2n}))')| \le \lambda_{p_1}(\varphi_{p_1}^{-1}(\lambda'_G)+\varphi_{p_1}^{-1}(\bar{f}_1)) + \lambda_{p_2}(\varphi_{p_2}^{-1}(\lambda'_G)+\varphi_{p_2}^{-1}(\bar{f}_2))$$

Similarly, we can find the same upper bound of $(m_1(T_1(u_{1n},u_{2n}))'$,thus

$$\{(m_1(T_1(u_{1n},u_{2n}))',m_2(T_2(u_{1n},u_{2n}))'\}$$

is bounded. Suppose that $h_i \in L^1(0,1)$, Since $\|m_i u'_{in}\|_\infty < \lambda_G$, then $|u'_{in}| < \lambda_G(m_i)^{-1} \in L^1(0,1)$. So $\{(u_{1n},u_{2n})\}$ is equicontinuous in $C[0,1]\times C[0,1]$ and by Arzela-Ascoli theorem, there exist a sequence $\{(u_{1nk},u_{2nk})\}$ of $\{(u_{1n},u_{2n})\}$ and $(v_1,v_2)\in C[0,1]\times C[0,1]$ such that $\{(u_{1nk},u_{2nk})\}\to(v_1,v_2)$.

Thus we have,

$$m_i(t)\varphi_{p_i}^{-1}(\varphi_{p_i}\left(\varphi_{p_i}u'_i\left(\tfrac{1}{2}\right)\right)\left(h_i f_i(u_{1nk},u_{2nk})\right) + \int_t^{\frac{1}{2}} h_i(s) f_i(u_{1nk}(s),u_{2nk}(s))\,ds) \to m_i(t)\varphi_{p_i}^{-1}(\varphi_{p_i}\left(\varphi_{p_i}u'_i\left(\tfrac{1}{2}\right)\right)\left(h_i f_i(v_1,v_2)\right) + \int_t^{\frac{1}{2}} h_i(s) f_i(v_1(s),v_2(s))\,ds), \quad (10)$$

Thus $\{(m_1(T_1(u_{1n},u_{2n}))', m_2(T_2(u_{1n},u_{2n}))'\}$ is equicontinuous.
If $h_i \in X_i \setminus L^1(0,1), \{(m_1(T_1(u_{1n},u_{2n}))',m_2(T_2(u_{1n},u_{2n}))'\}$ is not equicontinuous. Thus there exists $\varepsilon>0$ such that a sequence $\{(u_{1nk},u_{2nk})\}$ of $\{(u_{1n},u_{2n})\}$ and sequence $\{t_k\},\{s_k\}\subset(0,1)$ satisfying

$$|(m_i(T_i(u_{1nk},u_{2nk}))'(t_k) - m_i(T_i(u_{1nk},u_{2nk}))'(s_k)| \ge \varepsilon, \quad (11)$$

$|t_k - s_k| < \frac{1}{k}$, we show that $\lim_{k\to\infty} t_k = 0$ *or* 1. If it is not true, thus $\lim_{k\to\infty} t_k = t_0 \in (0,1)$. Let η satisfying $0<\eta_0< min\{t_0, 1-t_0\}$, so $h_i \in L^1[\eta_i, 1-\eta_i]$ and $u_{ink}\to v_i$. Then we prove that $\{m_i(T_i(u_{1n},u_{2n}))'\}$ is equicontinuous. So there is sufficiently large $N\in\mathbb{N}$ such that

$$|(m_i(T_i(u_{1nN},u_{2nN}))'(t_N) - m_i(T_i(u_{1nN},u_{2nN}))'(s_N)| < \varepsilon \quad (12)$$

and this contradicts with (11). Consider $\lim_{k\to\infty} t_k = \lim_{k\to\infty} s_k = 0$. $\lim_{k\to\infty} t_k = \lim_{k\to\infty} s_k = 1$ is similar. Then we get,

$$(m_i(T_i(u_{1nk},u_{2nk}))'(t_k) = \varphi_{p_i}^{-1}(m_i^{p_i-1}(t_k)\varphi_{p_i}\left(u'_i\left(\tfrac{1}{2}\right)\right)\left(h_i f_i(u_{1nk},u_{2nk})\right) + m_i^{p_i-1}(t_k)\int_t^{\frac{1}{2}} h_i(s) f_i\left(u_{1nk}(s),u_{2nk}(s)\right)ds, \quad (13)$$

So,

$$\lim_{k\to\infty} m_i^{p_i-1}(t_k)\varphi_{p_i}\left(u'_i\left(\tfrac{1}{2}\right)\right)\left(h_i f_i(u_{1nk},u_{2nk})\right) = 0 \quad (14)$$

Since $u_{ink}\to v_i$. we have

$$\lim_{k\to\infty} m_i^{p_i-1}(t_k)\int_{t_k}^{\frac{1}{2}} h_i(s)(f_i\left(u_{1nk}(s),u_{2nk}(s)\right) - (f_i\left(v_1(s),v_2(s)\right)ds \quad (15)$$

$\le \lim_{k\to\infty} m_i^{p_i-1}(t_k)\int_{t_k}^{\frac{1}{2}} h_i(s)ds\left\|f_i\left(u_{1nk}(s),u_{2nk}(s)\right) - (f_i\left(v_1(s),v_2(s)\right)\right\|_\infty$
$\lim_{k\to\infty}\left\|f_i\left(u_{1nk}(s),u_{2nk}(s)\right) - (f_i\left(v_1(s),v_2(s)\right)\right\|_\infty = 0.$
Now, we get

$$\lim_{k\to\infty} m_i^{p_i-1}(t_k)\int_{t_k}^{\frac{1}{2}} h_i(s) f_i\left(v_1(s),v_2(s)\right)ds = \begin{cases} f_i(0,0), h_i>0 \\ -f_i(0,0), h_i>0 \end{cases}$$

$m_i^{p_i-1}(t_k) = (\int_{t_k}^{\frac{1}{2}}|h_i(s)|\,ds)^{-1}$ implies that

$$\lim_{k\to\infty} m_i^{p_i-1}(t_k)\int_{t_k}^{\frac{1}{2}} h_i(s) f_i\left(v_1(s),v_2(s)\right)ds = \lim_{t\to0^+}\frac{\int_t^{\frac{1}{2}} h_i(s) f_i\left(v_1(s),v_2(s)\right)ds}{\int_t^{\frac{1}{2}}|h_i(s)|ds}, \quad (16)$$

for $h_i f_i(v_1,v_2)\in L^1(\left(0,\frac{1}{2}\right)\times\left(0,\frac{1}{2}\right))$ we have $f(0,0)=0$. $\lim_{t\to0^+}\int_{t_k}^{\frac{1}{2}}|h_i(s)|\,ds=\infty$ implies that the limit (16) is 0. $h_i(s)f_i\left(v_1(s),v_2(s)\right)\notin L^1\left[t,\frac{1}{2}\right]$.
Using L'Hospital's rule, we have

$$\lim_{k\to\infty}(m_i(T_i(u_{1nk},u_{2nk}))'(t_k) = \begin{cases} f_i(0,0), h_i>0 \\ -f_i(0,0), h_i>0 \end{cases} \quad (17)$$

$\lim_{k\to\infty}(m_i(T_i(u_{1nk},u_{2nk}))'(t_k) = 0$ and this contradicts with (11).
Thus $\{((m_1(T_1(u_{1nk},u_{2nk}))',(m_2(T_2(u_{1nk},u_{2nk}))'\}$ is equicontinuous in $C[0,1]\times C[0,1]$.
Suppose that $(u_{1n},u_{2n})\to(\ddot{u}_1,\ddot{u}_2)$ in K. Since $G_1\times G_2$ is compact, there is a sequence $\{(u_{1nj},u_{2nj})\}$ and $(v_1,v_2)\in K$ such that $(T_1(u_{1nk},u_{2nk}),(T_2(u_{1nk},u_{2nk}))\to(v_1,v_2)$
We Know that T is continuous and $(u_{1nj},u_{2nj})\to(\ddot{u}_1,\ddot{u}_2)$,
So $(T_1(u_{1nk},u_{2nk}),(T_2(u_{1nk},u_{2nk}))\to(T_1(\ddot{u}_1,\ddot{u}_2), T_2(\ddot{u}_1,\ddot{u}_2))$ then $(T_1(\ddot{u}_1,\ddot{u}_2), T_2(\ddot{u}_1,\ddot{u}_2))\equiv(v_1,v_2)$ thus T is continuous on K.

Definition (2) For $(\alpha_1,\alpha_2)\in C^2([0,1],\mathbb{R}), (\alpha_1,\alpha_2)$ is said to be a lower (strict lower) solution of

$$\begin{cases} \varphi_{p_1}(u_1') + F(u_1,u_2) = 0, t\in(0,1) \\ \varphi_{p_2}(u_2') + G(u_1,u_2) = 0, \\ u_1(0) = u_1(1) = u_2(0) = u_2(0). \end{cases}$$

$$\text{If } \begin{cases} \varphi_{p_1}(\alpha_1(t)')' + F\left(\alpha_1(t),\alpha_2(t)\right) \ge 0, (>), t\in(0,1) \\ \varphi_{p_2}(\alpha_2(t)') + G\left(\alpha_1(t),\alpha_2(t)\right) \ge 0,\ (>) \\ \alpha_1(0)\le 0\ (<0), \alpha_2(0)\le 0\ (<0), \\ \alpha_2(0)\le 0\ (<0), \alpha_2(1)\le 0\ (<0). \end{cases}$$

An upper (strict upper) solution $(\beta_1,\beta_2) \in C^2[0,1] \times C^2[0,1]$ can also be defined if it satisfies the reverse of the above inequalities.

RESULTS

Theorem (3) Suppose that there exist a strict lower solution (α_1,α_2) and a strict upper solution (β_1,β_2) of (1) such that $(\alpha_1,\alpha_2) < (\beta_1,\beta_2)$. Then problem (1) has at least one solution (u_1,u_2) such that $(\alpha_1,\alpha_2) < (u_1,u_2) < (\beta_1,\beta_2)$. Then the Leray-Schauder degree is $\deg(I-,\Omega,0) = 1$, (16) where,

$$\Omega = \{(u_1,u_2) \in K | (\alpha_1,\alpha_2) < (u_1,u_2) < (\beta_1,\beta_2), \|(u_1,u_2)\|_2 < r\}$$

for $r > 0$.

Proof. Let

$$\varphi_{p_1}(u_1'(t))' + h_1(t)f_1(\delta(u_1(t),u_2(t))) = 0 , t \in (0,1),$$
$$\varphi_{p_1}(u_1'(t))' + h_1(t)f_1(\delta(u_1(t),u_2(t))) = 0$$
$$u_1(0) = u_1(1) = u_2(0) = u_2(0) \tag{18}$$

Where $\delta: \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ is defined by

$$\delta(u_1,u_2) = \begin{cases} (\beta_1,\beta_2), & (u_1,u_2) > (\beta_1,\beta_2) \\ (u_1,u_2), (\alpha_1,\alpha_2) \le (u_1,u_2) \le (\beta_1,\beta_2), \\ (\alpha_1,\alpha_2), & (u_1,u_2) < (\alpha_1,\alpha_2) \end{cases} \tag{19}$$

If (u_1,u_2) be a solution of (18), then $(\alpha_1,\alpha_2) < (u_1,u_2) < (\beta_1,\beta_2)$ and (u_1,u_2) is a solution of (1). Let $\bar{T}: K \to K$ such that $\bar{T}(u_1,u_2) = T(\delta(u_1,u_2))$ so $\bar{T}$ is bounded and there exists $r \gg 1$ such that $\|\bar{T}(u_1,u_2)\|_2 <$r for $(u_1,u_2) \in K$. So we have,

$$\deg(I - \bar{T}, \mathrm{B_r}(0),0) = \deg(I, \mathrm{B_r}(0),0) = 1 \tag{20}$$

Where $\mathrm{B_r}(0) = \{(u_1,u_2) \in K | \ \|(u_1,u_2)\|_2 < r\}$. Thus (18) has a solution and (1) has a solution (u_1,u_2) satisfying $(\alpha_1,\alpha_2) < (u_1,u_2) < (\beta_1,\beta_2)$. From (20) we have

$$\deg(I - T, \Omega, 0) = \deg(I - \bar{T}, \Omega, 0)$$
$$= \deg(I, \mathrm{B_r}(0),0) = 1. \tag{21}$$

Theorem (4) Suppose that

$$(\alpha_1,\alpha_2) \le (\beta_1,\beta_2) \le (\gamma_1,\gamma_2) ,$$
$$(\alpha_1,\alpha_2) \le (\eta_1,\eta_2) \le (\gamma_1,\gamma_2) \tag{22}$$

where (α_1,α_2) is a lower solution , (γ_1,γ_2) an upper solution , (η_1,η_2) a strict lower solution and (β_1,β_2) a strict upper solution of (1) and there exists $t_0 \in [0,1]$ such that $(\beta_1(t_0),\beta_2(t_0)) < (\eta_1(t_0),\eta_2(t_0))$.

Then problem (1) has at least three solutions $(u_1,u_2),(u'_1,u'_2),(u''_1,u''_2)$ such that

$$(\alpha_1,\alpha_2) \le (u_1,u_2) < (\beta_1,\beta_2),$$
$$(\eta_1,\eta_2) < (u'_1,u'_2) \le (\gamma_1,\gamma_2) \tag{23}$$
$$(u''_1,u''_2) \in ([\alpha_1,\gamma_1] \times [\alpha_2,\gamma_2]) \setminus$$
$$(([\alpha_1,\beta_1] \times [\alpha_2,\beta_2]) \cup ([\eta_1,\gamma_1] \times [\eta_2,\gamma_2])).$$

Proof. Let

$$\varphi_{p_1}(u_1'(t))' + h_1(t)f_1(\delta(u_1(t),u_2(t))) = 0 , t \in (0,1),$$
$$\varphi_{p_1}(u_1'(t))' + h_1(t)f_1(\delta(u_1(t),u_2(t))) = 0$$
$$u_1(0) = u_1(1) = u_2(0) = u_2(0). \tag{24}$$

Where $\delta: \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ is defined by

$$\delta(u_1,u_2) = \begin{cases} (\gamma_1,\gamma_2), & (u_1,u_2) > (\gamma_1,\gamma_2) \\ (u_1,u_2), (\alpha_1,\alpha_2) \le (u_1,u_2) \le (\gamma_1,\gamma_2) \\ (\alpha_1,\alpha_2), & (u_1,u_2) < (\alpha_1,\alpha_2) \end{cases} \tag{25}$$

We know that $(\alpha_1 - \varepsilon, \alpha_2 + \varepsilon), (\gamma_1 + \varepsilon, \gamma_2 + \varepsilon)$ are strict lower solution and strict upper solution of (24). Thus, if (u_1,u_2) is a solution of (24), then

$$(\alpha_1 - \varepsilon, \alpha_2 + \varepsilon) < (\alpha_1,\alpha_2) \le (u_1,u_2)$$
$$\le (\gamma_1,\gamma_2) < (\gamma_1 + \varepsilon, \gamma_2 + \varepsilon).$$

Theorem 1, implies that there is a sufficient $r > 0$ such that the following satisfying,

$$\Omega_1 = \left\{(u_1,u_2) \in K \middle| \begin{matrix}(\alpha_1 - \varepsilon, \alpha_2 - \varepsilon) < (u_1,u_2) \\ < (\beta_1,\beta_2), \|(u_1,u_2)\|_2 < r\end{matrix}\right\}$$

$$\Omega_2 = \left\{(u_1,u_2) \in K \middle| \begin{matrix}(\eta_1,\eta_2) < (u_1,u_2) < \\ (\gamma_1 + \varepsilon, \gamma_2 + \varepsilon), \|(u_1,u_2)\|_2 < r\end{matrix}\right\}$$

$$\Omega_3 = \left\{(u_1,u_2) \in K \middle| \begin{matrix}(\alpha_1 - \varepsilon, \alpha_2 - \varepsilon) < (u_1,u_2) < \\ (\gamma_1 + \varepsilon, \gamma_2 + \varepsilon), \|(u_1,u_2)\|_2 < r\end{matrix}\right\}$$

let $T_\delta: K \to K$ is defined by $T_\delta(u_1,u_2)(t) = T(\delta(u_1,u_2))$. Thus we have $\deg(I - T_\delta, \Omega_3 \setminus (\overline{\Omega_1} \cup \overline{\Omega_2}) ,0) = -1$. So there are three solutions of (24), $(u_1,u_2) \in \overline{\Omega_1}$, $(u'_1,u'_2) \in \overline{\Omega_2}$, $(u''_1,u''_2) \in \Omega_3 \setminus (\overline{\Omega_1} \cup \overline{\Omega_2})$. Suppose that

$$\begin{cases} \varphi_{p_1}(u_1')' + \lambda_1 h_1(t) f_1(u_1,u_2) = 0, t \in (0,1) \\ \varphi_{p_2}(u_2')' + \lambda_2 h_2(t) f_2(u_1,u_2) = 0, \\ u_1(0) = u_1(1) = u_2(0) = u_2(0). \end{cases} \tag{26}$$

Where

$$\lambda_i > 0, h_i > 0,$$
$$H_1) \min_{t \in (0,1)} h_i(t) = \underline{h_i} > 0,$$
$$H_2) \lim_{u_1 \to \infty} \frac{f_1(u_1,0)}{\varphi_{p_1}(u_1)} = 0,$$
$$H_3) \lim_{u_2 \to \infty} \frac{f_2(0,u_2)}{\varphi_{p_2}(u_2)} = 0, \text{ and}$$
$$H_4) \ f_i \ \textit{is nondecreasing}.$$

Theorem (5) Suppose $(H_1) - (H_4)$ hold $a > 0, b > 0, c > 0, d > 0$ such that $a < b, c < d$ and $\frac{a^{p_1 - 1}}{f_{1(a,0)}} > r_1 \frac{b^{p_1-1}}{f_1(b,0)}$, $\frac{c^{p_2-1}}{f_{2(0,c)}} > r_2 \frac{d^{p_2-1}}{f_2(0,d)}$ where $r_i = 4^{p_i} (\frac{\|u_i\|_\infty^{p_i - 1}}{\underline{h_i}})$ and u_i the solution of

$$\varphi_{p_i}(u_i') + h_i(t) = 0 , \quad u_i(0) = u_i(1) = 0 \tag{27}$$

then for $\theta_i > 0$ satisfying

$$r_1\frac{b^{p_1-1}}{f_1(b,0)\|u_1\|_\infty^{p_1-1}} < \theta_1 < \frac{a^{p_1-1}}{f_{1(a,0)}\|u_1\|_\infty^{p_1-1}} ,$$

$$r_2\frac{d^{p_2-1}}{f_2(0,d)\|u_2\|_\infty^{p_2-1}} < \theta_2 < \frac{c^{p_2-1}}{f_{2(0,c)}\|u_2\|_\infty^{p_2-1}} , \tag{28}$$

problem (26) has three positive solutions.

Proof. From (28) it is easy to see that $(\alpha_1,\alpha_2)\equiv(0,0)$ is a lower solution of (26).
Suppose $(\beta_1,\beta_2)=(a\left(\frac{u_1}{\|u_1\|_\infty}\right),c\left(\frac{u_2}{\|u_2\|_\infty}\right))$. We have

$$-\varphi_{p_1}\left(\beta'_1(t)\right)'=\frac{a^{p_1-1}}{\|u_1\|_\infty^{p_1-1}}\varphi_{p_1}(u_1')'=\frac{a^{p_1-1}}{\|u_1\|_\infty^{p_1-1}}h_i(t) > \theta_1 h_1(t) f_1(a,0)\geq \theta_1 h_1(t) f_1(\beta_1(t),0). \tag{29}$$

Thus $(\beta_1,\beta_2)=(a\left(\frac{u_1}{\|u_1\|_\infty}\right),c\left(\frac{u_2}{\|u_2\|_\infty}\right))$ is an upper solution of (26). Now, let (u_1,u_2) be a solution of (26) and $(u_1,u_2)\leq(\beta_1,\beta_2)=(a\left(\frac{u_1}{\|u_1\|_\infty}\right),c\left(\frac{u_2}{\|u_2\|_\infty}\right))$, we show that $(u_1,u_2)<(\beta_1,\beta_2)$. Otherwise, there exist $t_1<t_2$ such that $(u_1'(t_2),u_2'(t_2))<(\beta_1'(t_2),\beta_2'(t_2))$ and $(u_1'(t_1),u_2'(t_1))=(\beta_1'(t_1),\beta_2'(t_1))$
Then we have

$$0\leq\int_{t_1}^{t_2}-\theta_i h_i(s) f_i(u_1(s),u_2(s)) + \theta_i h_i(s) f_i(\beta_1(s),\beta_2(s)) < \int_{t_1}^{t_2}\varphi_{p_i}\left(u_i'(s)\right)' - \varphi_{p_i}\left(\beta_i'(s)\right)' ds<0, \tag{30}$$

It is a contradiction. Since

$$\varphi_{p_i}\left(u_i'(t)\right)' - \varphi_{p_i}\left(\beta_i'(t)\right)' >0, t\in(0,1), \tag{31}$$

There is $e\in(0,1)$ such that

$(u_1'(e),u_2'(e))>(\beta_1'(e),\beta_2'(e))$ if not $(u_1'(e),u_2'(e))\leq(\beta_1'(e),\beta_2'(e))$ so we have $\varphi_{p_i}(u_i'(1))-\varphi_{p_i}(\beta_i'(1))>\varphi_{p_i}(u_i'(d))-\varphi_{p_i}(\beta_i'(d))>0$ and thus $m_i u_i'(1)>m_i\beta_i'(1)$ then

$$m_i\beta_i'(1)=\lim_{t\to1^-}m_i(t)\beta_i'(t)=$$

$$\lim_{t\to1^-}\frac{1}{\varphi_{p_i}^{-1}(\int_{\frac12}^t h_i(s)ds)}\times\left[\begin{array}{c}-\varphi_{p_i}^{-1}(-\alpha_i\left(\frac{a^{p_1-1}}{\|u_1\|_\infty^{p_1-1}}h_i\right)+\\ \frac{a^{p_1-1}}{\|u_1\|_\infty^{p_1-1}}\int_{\frac12}^t h_i(s)ds)\end{array}\right]=\varphi_{p_i}^{-1}\left(-\frac{a^{p_1-1}}{\|u_1\|_\infty^{p_1-1}}\right).$$

L'Hospital's rule, implies that

$$m_i u_i'(1)=\lim_{t\to1^-}m_i(t)u_i'(t)=$$

$$\lim_{t\to1^-}\frac{1}{\varphi_{p_i}^{-1}(\int_{\frac12}^t h_i(s)ds)}\times\left[\begin{array}{c}-\varphi_{p_i}^{-1}(-\alpha_i\left(\theta_i h_i f_i(u_1,u_2)\right)+\\ \int_{\frac12}^t \theta_i h_i(s) f_i(u_1(s),u_2(s))ds)\end{array}\right]$$

$$=\lim_{t\to1^-}-\varphi_{p_i}^{-1}(-\frac{\alpha_i\left(\theta_i h_i f_i(u_1,u_2)\right)}{\int_{\frac12}^t h_i(s)ds}+\frac{\int_{\frac12}^t\theta_i h_i f_i(u_1,u_2)ds}{\int_{\frac12}^t h_i(s)ds})$$

$$=\varphi_{p_i}^{-1}(-\lim_{t\to1^-}\frac{\int_{\frac12}^t\theta_i h_i(s) f_i(u_1(s),u_2(s))ds}{\int_{\frac12}^t h_i(s)ds})=\varphi_{p_i}^{-1}(-\lim_{t\to1^-}\frac{\theta_i h_i(t) f_i(u_1(t),u_2(t))}{h_i(t)})$$

$$=\varphi_{p_i}^{-1}\left(-\lim_{t\to1^-}\theta_i f_i(u_1(t),u_2(t))\right)=\varphi_{p_i}^{-1}(-\theta_i f_i(0,0))$$

For i=1 we have $m_1u_1'(1)>\varphi_{p_1}^{-1}(-\frac{a^{p_1-1}f_1(0,0)}{f_{1(a,0)}\|u_1\|_\infty^{p_1-1}})$

$$>\varphi_{p_1}^{-1}\left(-\frac{a^{p_1-1}}{\|u_1\|_\infty^{p_1-1}}\right)=m_1\beta_1'(1)$$

Thus $m_1u_1'(1)>m_1\beta_1'(1)$ we proved (β_1,β_2) is a strict upper solution of (26).
Now, let θ_1^* , k, j>1 satisfying $r_1\frac{b^{p_1-1}}{f_1(b,0)\|u_1\|_\infty^{p_1-1}}<\theta_1^*<\theta_1$, $1<(k_1j)^{p_1-1}<\frac{\theta_1^*\underline{h_1}f_1(b,0)}{4^{p_1}b^{p_1-1}}$.

Suppose that η_1 be the solution of

$$\varphi_{p_1}\left(\eta_1'(t)\right)'+\theta_1^*\underline{h_1}f_1(v_1(t),v_2(t))=0,$$
$$\eta_1(0)=\eta_1(1)=0,$$

Where $v_1(t)=b\tau_1(t)$, $v_2(t)=d\tau_2(t)$, and

$$\tau_i(t)=\begin{cases}1-(1-(4t)^{k_i})^j, & 0\leq t\leq\frac14\\ 1 & ,\frac14\leq t\leq\frac12\end{cases}$$

So we have

$$\eta_1'(t)=\varphi_{p_1}^{-1}(\int_t^{\frac12}\theta_1^*\underline{h_1}f_1(v_1(s),v_2(s))ds$$

$$\geq\varphi_{p_1}^{-1}(\int_{\frac14}^{\frac12}\theta_1^*\underline{h_1}f_1(m_1(s),m_2(s))ds$$

$$=(\theta_1^*\underline{h_1}f_1(b,0)\frac14)^{\frac{1}{p_1-1}}>v_1'(t).$$

then $(\eta_1,\eta_2)>(v_1,v_2)$, we can see that

$$m_i u_i'(0) = \lim_{t\to 0^+} m_i(t)u_i'(t) = \lim_{t\to 0^+} \frac{1}{\varphi_{p_i}{}^{-1}(\int_t^{\frac{1}{2}} h_i(s)ds)}$$
$$\times \left[\varphi_{p_i}{}^{-1}(\alpha_i(\theta_i h_i f_i(u_1,u_2)) + \int_t^{\frac{1}{2}} \theta_i h_i(s) f_i(u_1(s),u_2(s))ds) \right]$$
$$= \varphi_{p_i}{}^{-1}(\lim_{t\to 0^+} \frac{\theta_i h_i(t) f_i(u_1(t),u_2(t))}{h_i(t)})$$
$$= \theta_i f_i(0,0) > 0 = m_i \eta_i'(0).$$

We conclude that $(u_1,u_2) \geq (\eta_1,\eta_2)$. Thus (η_1,η_2) is a strict lower solution of (26).

Let $\gamma_1 = \frac{\theta_1 l_1 u_1}{\|u_1\|_\infty}$, $\gamma_2 = \frac{\theta_2 l_2 u_2}{\|u_2\|_\infty}$, From (H_3) there exists $l_i \gg 1$ such that $\frac{f_i(\theta_1 l_1,\theta_2 l_2)}{(\theta_i l_i)^{p_i-1}} < \frac{\theta_i}{\|u_i\|_\infty^{p_i-1}}$,

$(\eta_1,\eta_2) > (\gamma_1,\gamma_2)$, $(\beta_1,\beta_2) < (\gamma_1,\gamma_2)$.

So

$$-\varphi_{p_i}(\gamma_i'(t))' = -\frac{(\theta_i l_i)^{p_i-1}\varphi_{p_i}(u_i'(t))'}{\|u_i\|_\infty^{p_i-1}} >$$
$$\theta_i h_i(t) f_i(\theta_1 l_1,\theta_2 l_2) \geq \theta_i h_i(t) f_i(\gamma_1(t),\gamma_2(t)).$$

Thus (γ_1,γ_2) is an upper solution of (26) and by theorem 4 problem (1) has three solutions.

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