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Behavior of elliptical objects in general theory of relativity

Bijan Nikouravan[1,2,3] , J. J. Rawal [3]

[1]Department of Physics, University of Malaya, 50603 Kuala Lumpur, Malaysia
[2] Department of Physics, Islamic Azad University (IAU)-Varamin-Pishva Branch, Iran
Email: nikou@um.edu.my , nikouravan@gmail.com
[3] The Indian Planetary Society (IPS), Mumbai - 400091, India
Email: ips.science@gmail.com

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ABSTRACT

The simplest solution to Einstein's field equations is the Schwarzschild solution. This solution is not able to describe any non-spherical shaped objects. Some stars and galaxies are ellipsoidal. Consequently, the gravitational field around these objects should be different in comparison with the spherical form. This paper is considering a new line element so that we are able to construct not only spherical objects but also we are able to explain an ellipsoidal object too. This new line element is more accurate and complete than the Schwarzschild line element. In this research, we see that the Schwarzschild line element and its solution is only a part of the whole work, which we have done. For more consideration, we applied this metric to an arbitrary object in the next step. Moreover, we used this line element for the solution of a planetary orbit of an ellipsoid planet by using Einstein's field equations. These equations used for the exterior solution of an ellipsoidal celestial object.

Key words: General relativity, elliptical objects, planetary orbits

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INTRODUCTION

Gravitational field equations described by general theory of relativity (Einstein, 1916). These equations are able to explain the properties of gravitational field around celestial objects. The first person who applied these equations was the German astrophysicist Karl Schwarzschild (Schwarzschild, 1916). He solved these equations for the first time and described the gravitational fields only around spherically symmetric and non-rotating objects in the static form. The exact solution to the Einstein field equations is the Schwarzschild metric. This solution is corresponding to the external gravitational field of a stationary and uncharged object. He ignored the effects of the star's interior in his solution. However, Schwarzschild solution is the simplest solution of Einstein's field equations, but it is not able to describe non-spherical objects such as elliptical objects like stars and /or galaxies. The simple structures of elliptical galaxies are reflected in their place in Hubble's Classification. They are characterized by a single number, the ellipticity $\varepsilon = 10(1 - b/a)$, where b and a are the projected angular extent of the short and long axis of the galaxy on the sky (Roger, 2006). In fact, most of the celestial objects like stars and planets are not exactly spherical but fairly ellipsoidal in shape. Therefore, the Schwarzschild solution is unsuitable for elliptical objects in shape. Certainly, to obtain the gravitational field around these types of objects, we need some more modification in our metric and line element too. The purpose of the present paper is to construct a framework for considering ellipsoidal shapes in general theory of relativity, which covers situations studied for all ellipsoidal objects.

ELLIPTICAL OBJECTS

In a good approximation huge bodies like galaxies and stars-cluster, are in the static form. Therefore, assumed rotation of these objects may not play an important role in our study and for general purpose. Some of these objects are in the form elliptic and In the Euclidean geometry, the concept of an ellipsoid object, completely is clear but in the curved spaces, it has some different meaning. Since, the geometry of General Theory of Relativity (GTR) is based on Riemannian geometry, therefore, the curved space and its analysis is necessary. In this case, the perfect-fluid bodies, having an ellipsoidal shape and it is esential to determine the curvature of space and time in the presence of an ellipsoid (Zsigrai 2008). We, therefore, try to find out a solution to the Einstein's field equations for static and ellipsoidal shaped heavenly bodies where we are not much concerned with its rotation. In the absence of any mass point, the space time would be flat. Consider ordinary Minkowski space-time, described by the coordinates (x, y, z), where the static line element is defined as:

$$ds^2 = c^2dt^2 + dx^2 - dy^2 - dz^2 \tag{1}$$

By performing the following coordinate transformations, (Landau and Lifshitz, 1987)

$$\begin{aligned} &x \to (r^2 + a^2)^{\frac{1}{2}} \sin\theta \cos\varphi \\ &y \to (r^2 + a^2)^{1/2} \sin\theta \sin\varphi \\ &z \to r\cos\theta \quad , \quad t \to t \end{aligned} \tag{2}$$

on the metric given in (1), the metric in the new coordinate (Nikouravan, 2001) is,

$$ds^2 = c^2dt^2 - \frac{r^2 + a^2cos^2\theta}{r^2 + a^2}dr^2 - (r^2 + cos^2\theta)d\theta^2 + (r^2 + a^2)sin^2\theta d\varphi^2 \tag{3}$$

In this coordinate frame, a is a constant in the, xy surface. The above line element (3) is valid only for a vacuum space and time. This line element in the presence of a mass point takes the following form

$$ds^2 = e^{\nu}dt^2 - e^{\lambda}\left(\frac{r^2 + a^2cos^2\theta}{r^2 + a^2}\right)dr^2 - (r^2 + cos^2\theta)d\theta^2 + (r^2 + a^2)sin^2\theta d\varphi^2 \tag{4}$$

Here e^{ν} and e^{λ} are as coefficient the parameters λ and ν, are function of r and θ only and c is the velocity of light and supposed as unit $(c = 1)$. By using covariant, g_{ij} and contravariant, g^{ij} components $(i, j = 1,2,3,4)$ of the metric tensors (4) the values of non-vanishing first kinds of Christofell's symbols for are;

$$\begin{aligned}
&\Gamma_{1/11} = e^{\lambda}\left[\frac{\lambda'(r^2 + a^2cos^2\theta)}{2(r^2 + a^2)} + \frac{r}{(r^2 + a^2)} + \frac{r(r^2 + a^2cos^2\theta)}{(r^2 + a^2)^2}\right] \\
&\Gamma_{3/23} = \Gamma_{3/32} = \frac{1}{2}(r^2 + a^2)\sin 2\theta \\
&\Gamma_{1/12} = \Gamma_{1/21} = -e^{-\lambda}\left[\frac{\dot\lambda(r^2 + a^2cos^2\theta)}{2(r^2 + a^2)} + \frac{a^2\, sin2\theta}{2(r^2 + a^2)}\right] \\
&\Gamma_{2/11} = e^{\lambda}\left[\frac{\dot\lambda(r^2 + a^2cos^2\theta)}{2(r^2 + a^2)} - \frac{a^2\, sin2\theta}{2(r^2 + a^2)}\right] \\
&\Gamma_{2/12} = \Gamma_{2/21} = -r \quad , \quad \Gamma_{2/33} = \frac{1}{2}(r^2 + a^2)sin2\theta \\
&\Gamma_{2/44} = -e^{\nu}\left(\frac{\dot\nu}{2}\right), \qquad \Gamma_{1/44} = -e^{\nu}\left(\frac{\nu'}{2}\right) \\
&\Gamma_{3/13} = -r\,sin^2\theta \ , \ \Gamma_{1/22} = r \ , \ \Gamma_{1/33} = r\,sin^2\theta \\
&\Gamma_{4/14} = \Gamma_{4/41} = e^{\nu}\left(\frac{\nu'}{2}\right), \ \Gamma_{4/24} = \Gamma_{4/42} = e^{\nu}\left(\frac{\dot\nu}{2}\right)
\end{aligned} \tag{5}$$

Also the second kinds of Christofell's symbols are as,

$$\begin{aligned}
&\Gamma^1_{11} = \frac{\lambda'}{2} + \frac{r}{(r^2 + a^2\cos^2\theta)} - \frac{r}{(r^2 + a^2)} \\
&\Gamma^1_{12} = \Gamma^1_{21} = \frac{\dot\lambda}{2} - \frac{a^2\sin 2\theta}{2(r^2 + a^2\cos^2\theta)} \\
&\Gamma^1_{22} = \frac{-r(r^2 + a^2)}{e^{\lambda}(r^2 + a^2\cos^2\theta)} \ , \\
&\Gamma^1_{33} = \frac{-r(r^2 + a^2)\sin^2\theta}{e^{\lambda}(r^2 + a^2\cos^2\theta)} \ , \\
&\Gamma^1_{44} = \frac{\upsilon' e^{\upsilon}(r^2 + a^2)}{2e^{\lambda}(r^2 + a^2\cos^2\theta)} \ , \\
&\Gamma^2_{44} = \frac{\dot\upsilon e^{\upsilon}}{2(r^2 + a^2\cos^2\theta)} \ , \\
&\Gamma^2_{33} = \frac{-(r^2 + a^2)\sin 2\theta}{2(r^2 + a^2\cos^2\theta)} \ , \\
&\Gamma^2_{12} = \Gamma^2_{21} = \frac{r}{(r^2 + a^2\cos^2\theta)} \ , \\
&\Gamma^2_{11} = \frac{-\dot\lambda e^{\lambda}}{2(r^2 + a^2)} + \frac{e^{\lambda}a^2\sin 2\theta}{2(r^2 + a^2)(r^2 + a^2\cos^2\theta)} \ , \\
&\Gamma^3_{13} = \Gamma^3_{31} = \frac{r}{(r^2 + a^2)} \ , \ \Gamma^3_{23} = \Gamma^3_{32} = \mathrm{cotg}\,\theta \ , \\
&\Gamma^4_{14} = \Gamma^4_{41} = \frac{\upsilon'}{2} \ , \ \Gamma^4_{24} = \Gamma^4_{42} = \frac{\dot\upsilon}{2}
\end{aligned} \tag{6}$$

Here $\lambda' = \frac{\partial\lambda}{\partial r}$, $\dot\lambda = \frac{\partial\lambda}{\partial\theta}$, $\upsilon' = \frac{\partial\upsilon}{\partial r}$ and $\dot\upsilon = \frac{\partial\upsilon}{\partial\theta}$ have their usual meaning. Consequently different values of Ricci tensors are:

$$\begin{aligned}
R_{11} = \frac{\upsilon''}{2} + \frac{\upsilon'^2}{4} - \frac{\lambda'\upsilon'}{4} - \frac{\lambda'}{r} &+ \left(\frac{e^{\lambda}}{2r^2}\right)\left[\ddot\lambda + \frac{\dot\lambda^2}{2} + \frac{\dot\lambda\dot\upsilon}{2} + \dot\lambda\, cotg\theta\right] \\
&+ [\text{small terms } \mathrm{a}^2, \mathrm{a}^3 \text{or}, \mathrm{a}^4]
\end{aligned} \tag{7}$$

$$R_{22} = -1 + e^{-\lambda} + re^{-\lambda}\frac{(\upsilon' - \lambda')}{2} + \frac{\ddot\lambda}{2} + \frac{\ddot\upsilon}{2} + \frac{(\dot\lambda^2 + \dot\upsilon^2)}{4} + [\text{small terms } \mathrm{a}^2, \mathrm{a}^3 \text{or}, \mathrm{a}^4] \tag{8}$$

$$R_{33} = Sin^2\theta\left[-1 + e^{-\lambda} + re^{-\lambda}\frac{(\upsilon'-\lambda')}{2} + \left(\frac{\dot{\lambda}+\dot{\upsilon}}{2}\right)cotg\theta\right] \quad (9)$$

$$R_{44} = -\left(e^{\upsilon-\lambda}\right)\left[\frac{\upsilon''}{2} + \frac{\upsilon'^2}{4} - \frac{\lambda'\upsilon'}{4} + \frac{\upsilon'}{r}\right] - \left(\frac{e^{\upsilon}}{2r^2}\right)\left[\ddot{\upsilon} + \frac{\dot{\upsilon}^2}{2} + \frac{\lambda^o\upsilon^o}{2} + \upsilon^o \mathrm{cotg}\theta\right] \quad (10)$$

With *Mathematica*, we are able to calculate all components of Ricci tensors (Nikouravan 2009). All components of Ricci tensors R_{ij}'s, (7), (8), (9) and, (10) for θ approximately constant, are identically to zero and simplifies to the following form,

$$R_{11} = \frac{\upsilon''}{2} + \frac{\upsilon'^2}{4} - \frac{\lambda'\upsilon'}{4} - \frac{\lambda'}{r} \quad (11)$$

$$R_{22} = e^{-\lambda}\left(\frac{\upsilon' r}{2} - \frac{\lambda' r}{2} + 1\right) - 1 \quad (12)$$

$$R_{33} = \left[e^{-\lambda}\left(\frac{\upsilon' r}{2} - \frac{\lambda' r}{2} + 1\right) - 1\right]\sin^2\theta \quad (13)$$

$$R_{44} = -\frac{1}{2}e^{\upsilon-\lambda}\left(\upsilon'' + \frac{\upsilon'^2}{2} - \frac{\lambda'\upsilon'}{2} + \frac{2\upsilon'}{r}\right) \quad (14)$$

The solution of these equations (11), (12), (13), and (14) for λ and ν we get, $r^{-1}(\partial\upsilon/\partial r) + (\partial\lambda/r\partial r) = 0$. After integration we have $\lambda + \upsilon = A$. The value of A is a constant of integration which may be set equal to zero. For large r, the values $\lambda = 0$ and $\upsilon = 0$ then $\lambda = -\upsilon$ By substituting in the above equations we get, $e^{\upsilon}\left(1 + \frac{r\partial\upsilon'}{\partial r}\right) = 1$. After integrating we have $re^{\upsilon} = r + B$. Here the value of B being constant of integration *i.e.* $e^{\upsilon} = e^{-\lambda} = 1 - 2m/r$, where we have put $B = -2m$. This is done in order to facilitate the physical interpretation of m as the mass of the gravitating particle. Finally, the suggested line element due to a static and ellipsoidal isolated gravitating mass point get in the following form (Nikouravan, 2011).

$$ds^2 = \left(1 - \frac{2m}{r}\right)dt^2 - \frac{1}{\left(1 - \frac{2m}{r}\right)}\left(\frac{r^2 + a^2cos^2\theta}{r^2 + a^2}\right)dr^2 - (r^2 + cos^2\theta)d\theta^2 - (r^2 + a^2)sin^2\theta\, d\varphi^2 \quad (15)$$

Here the meaning of m is as Schwarzschild equation. Indeed, if we set $m = GM/c^2$ then we see that, for large value r $g \approx GM/r^2$ so that M is the mass of central body. In relativistic units $(c = G = 1)$ we simply have $m = M$ (Wolfgang, 2006) and dimensionally are correct. In equation (16), if we put $a = 0$ then it become Schwarzschild line element. In fact Schwarzschild metric is only a spherically symmetric solution of Einstein's equation for the vacuum space (Chandrasekhar, 1983).

$$ds^2 = \left(1 - \frac{2m}{r}\right)dt^2 - \frac{1}{\left(1 - \frac{2m}{r}\right)}dr^2 - r^2d\theta^2 - r^2sin^2\theta\, d\varphi^2 \quad (16)$$

Equation (15) is more general and complete as compared with equation (16). The line element (15) is not only valid for ellipsoidal form of any object but also it can explain the spherical form of object too.

PLANETARY ORBITS

Here we consider a solution for planetary orbits by using Einstein's field equations (Einstein, 1916). In terms of curved coordinate system x^i, we start with the line element (15) and attempt to solve these equations as exterior solution for a planet going around an ellipsoidal star. Indeed, we need to have equations that determine the connection field surrounding a heavy elliptical object such that it describes the gravitational field correctly (Hooft 2009). Consequently it is assumed that the ellipsoidal star remains at the center and the planet is rotating around the star. Therefore, the geodesic differential equations of the ellipsoid and their space-time trajectories are given by,

$$\frac{dx^\alpha}{ds^2} + \Gamma^\alpha_{\mu\upsilon}\frac{dx^\mu}{ds}\frac{dx^\upsilon}{ds} = 0 \quad (17)$$

In the line element (15), m and r, are the mass and radius of ellipsoidal star, respectively. By using (6) for the non-vanishing Christofell's symbols of second kind, and (18), we have four differential equation of motion as below.

$$\frac{d^2r}{ds^2} + \left[\Gamma^1_{11}\left(\frac{dr}{ds}\right)^2 + \Gamma^1_{22}\left(\frac{d\theta}{ds}\right)^2 + \Gamma^1_{33}\left(\frac{d\phi}{ds}\right)^2 + \Gamma^1_{44}\left(\frac{dt}{ds}\right)^2 + 2\Gamma^1_{12}\left(\frac{dr}{ds}\frac{d\theta}{ds}\right)\right] = 0 \quad (18)$$

$$\frac{d^2\theta}{ds^2} + e^{\lambda}\left[\Gamma^2_{11}\left(\frac{dr}{ds}\right)^2 + \Gamma^2_{22}\left(\frac{d\theta}{ds}\right)^2 + \Gamma^2_{33}\left(\frac{d\phi}{ds}\right)^2 + \Gamma^2_{44}\left(\frac{dt}{ds}\right)^2 + 2\Gamma^2_{12}\left(\frac{dr}{ds}\frac{d\theta}{ds}\right)\right] = 0 \quad (19)$$

$$\frac{d^2\phi}{ds^2} + \frac{2r}{(r^2 + a^2)}\left(\frac{dr}{ds}\frac{d\phi}{ds}\right) + 2\cot g\theta\left(\frac{d\theta}{ds}\frac{d\phi}{ds}\right) = 0 \quad (20)$$

$$\frac{d^2t}{ds^2} + \upsilon'\left(\frac{dr}{ds}\frac{dt}{ds}\right) + \dot{\upsilon}\left(\frac{d\theta}{ds}\frac{dt}{ds}\right) = 0 \quad (21)$$

The above relations are the equations of motion of a secondary going around an ellipsoidal star. If the planet moves initially in a plane $\theta = \frac{\pi}{2}$ then $\frac{d\theta}{ds} = 0$, and the above equations are,

$$\frac{d^2r}{ds^2} + \frac{\lambda'}{2}\left(\frac{dr}{ds}\right)^2 - re^{-\lambda}\sin^2\theta\left(\frac{d\phi}{ds}\right)^2 + e^{\upsilon-\lambda}\frac{\upsilon'}{2}\left(\frac{dt}{ds}\right)^2 = 0 \quad (22)$$

$$-e^{\lambda}\left(\frac{\dot{\lambda}}{2r^2}\right)\left(\frac{dr}{ds}\right)^2 + e^{\upsilon}\left(\frac{\dot{\upsilon}}{2r^2}\right)\left(\frac{dt}{ds}\right) = 0 \quad (23)$$

$$\frac{d^2\phi}{ds^2}+\frac{2r}{(r^2+a^2)}\left(\frac{dr}{ds}\frac{d\phi}{ds}\right)=0 \tag{24}$$

$$\frac{d^2t}{ds^2}+\upsilon'\left(\frac{dr}{ds}\frac{dt}{ds}\right)=0 \tag{25}$$

The solution of these equations yields,

$$\begin{cases}\dfrac{1}{(r^2+a^2)}\dfrac{d}{ds}\left((r^2+a^2)\dfrac{d\phi}{ds}\right)=0 & i.e.\ \dfrac{d}{ds}\left((r^2+a^2)\dfrac{d\phi}{ds}\right)=0\\ \dfrac{1}{e^{\upsilon}}\dfrac{d}{ds}(e^{\upsilon}\dfrac{dt}{ds})=0 & i.e.\ \dfrac{d}{ds}(e^{\upsilon}\dfrac{dt}{ds})=0\end{cases}$$

By integrating the above equations we have,

$$(r^2+a^2)\frac{d\phi}{ds}=h \ \ , \ \ e^{\upsilon}\frac{dt}{ds}=k\ , \tag{26}$$

Where h and k are constants of integration. The constant h is a measure of the angular momentum of the motion. Further, instead of working with equation (24) due to its troublesome integration, we use the line element (4). By using $\frac{d\theta}{ds}=0$, $\lambda=-\upsilon$, h and k as constant of integration, we get,

$$\left(\frac{r^2}{r^2+a^2}\right)\left(\frac{dr}{ds}\right)^2+e^{\upsilon}\frac{h^2}{(r^2+a^2)}-k^2+e^{\upsilon}=0 \tag{27}$$

Using $(dr/ds)=(dr/d\phi)(d\phi/ds)=(h/r^2)(dr/d\phi)$ and $e^{v}=1-2m/r$ the equation (27) becomes

$$\left(\frac{r^2h^2}{(r^2+a^2)^3}\right)\left(\frac{dr}{d\phi}\right)^2+\left(1-\frac{2m}{r}\right)\left[1+\frac{h^2}{(r^2+a^2)}\right]-k^2=0 \tag{28}$$

Now, if we substitute $u=1/r$ in the above equation and after rearranging, we get

$$\left(\frac{du}{d\phi}\right)^2+\left(1+a^2u^2\right)^2u^2=\left(\frac{k^2-1}{h^2}\right)\left(1+a^2u^2\right)^3+\left(\frac{2mu}{h^2}\right)\left(1+a^2u^2\right)^3+2mu^3\left(1+a^2u^2\right)^2 \tag{29}$$

Differentiating the above equation with respect to φ we can easily get

$$\frac{d^2u}{d\phi^2}+u\left(1+a^2u^2\right)\left(1+3a^2u^2\right)=\left(\frac{m}{h^2}\right)\left(1+a^2u^2\right)^3+mu^2\left(1+a^2u^2\right)\left(3+7a^2u^2\right)+3a^2u\left(\frac{k^2-1}{h^2}\right)\left(1+a^2u^2\right)^2+\left(\frac{6ma^2u^2}{h^2}\right)\left(1+a^2u^2\right)^2 \tag{30}$$

Equation (30) represents the relativistic differential equation of the path of a planet going around an ellipsoidal star. Here r and φ are the special coordinates, and ds is an element of the proper time as measured by a clock moving with the planet. In approximation small terms multiplied with u^3 and greater powers of u in equation (30) are still small and hence can be approximated to zero and get,

$$\left(\frac{d^2u}{d\phi^2}\right)+u\left(1-\frac{3a^2(K^2-1)}{h^2}\right)=\frac{m}{h^2}+3mu^2\left(1+\frac{2a^2}{h^2}\right) \tag{31}$$

Therefore, the relativistic differential equation of the orbit of the planet, equation (31), can be compared with the corresponding Schwarzschild line element for a spherical planet which is as below.

$$\frac{d^2u}{d\phi^2}+u=\left(\frac{m}{h^2}\right)+3mu^2 \tag{32}$$

and Newtonian equation, (Ramsey,1961)

$$\left(\frac{d^2u}{d\phi^2}\right)+u=\frac{m}{h^2} \tag{33}$$

RESULTS AND DISCUSSION

By comparing line elements (4) and (15) we get the values of e^{λ} and e^{v} are in terms of mass of the object or mass of the gravitating particle, like Schwarzschild. The relation between these two factors, is $e^{\upsilon}=e^{-\lambda}=1-2m/r$. By applying these values, we get the line element in the form of (17) as follows,

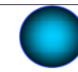
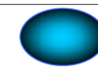
$$ds^2=(1-\frac{2m}{r})dt^2-\frac{1}{(1-\frac{2m}{r})}\left[\frac{r^2+a^2\cos^2\theta}{r^2+a^2}\right]dr^2-(r^2+a^2\cos^2\theta)d\theta-(r^2+a^2)\sin^2\theta d\phi^2$$

The line element (15) is more complete than Schwarzschild line element (16). The line element (16) can be obtained by substituting $a=0$ in (15). But the main difference between (15) and (16) is the value of a. For an ellipsoid, $a\neq 0$ and for a spherical object $a=0$. The equation (16) is valid only for spherical and is not possible to find out any non-spherical line element. The lines of gravitational field around the spherical and elliptical objects certainly are in different form. Therefore, the motion of secondary around the first object, in case of elliptical and/or spherical, certainly should be different. Hence the equations of motion, for elliptical in shape, is calculated using general theory of relativity and the result is (31).

The equation (31) simply shows the effect of elliptical objects in the space and also is the differential equation of motion. By substituting, $a=0$ in (31), we get (32), which describes the same conditions for the Schwarzschild line element. It means, Schwarzschild line element (16) and it's relating calculated equations of motion, for planetary orbit (32), all are results of (15) and (31), respectively. Certainly the Newtonian differential equation of motion (33) also is the next result of equation (15) and (31). The pictures given below show the results of calculations and differential equations of motion for flat, spherical and elliptical objects separately.

CONCLUSION

The aim of this work was to obtain a new form of line element which should be able to describe gravitational field around an elliptical object with solution of general theory of relativity.
This solution not only describes the gravitational field around elliptical objects in shape but also it can explain the field around spherical objects too. One of the applications of this line element is planetary orbit of an object around the elliptical object. Differential equations of motion for planetary orbits in elliptical and spherical are different. For elliptical objects we found a new term as mentioned in the equation (31).

	Geometry of space	Differential equation of motion	
1	Newtonian equation	$\left(\frac{d^2u}{d\phi^2}\right)+u=\frac{m}{h^2}$	
2	Spherical Schwarzschild	$\left(\frac{d^2u}{d\phi^2}\right)+u=\frac{m}{h^2}+3mu^2$	
3	Elliptical	$\left(\frac{d^2u}{d\phi^2}\right)+u\left(1-\frac{3a^2(k^2-1)}{h^2}\right)=\frac{m}{h^2}+3mu^2\left(1+\frac{2a^2}{h^2}\right)$	

The differential equation of motion of elliptical objects for ($a=0$) is same as differential equation of motion for spherical objects. Consequently, elliptical line element (16) and differential equation of motion (31) are more general and accurate than spherical form.

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